Endogenous Networks with Defaults, and Systemic Risk

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Abstract

Using random network applications and simulations, this paper analyzes the possible contagion of financial endogenous defaults through the network. We study (i) the effect of the network on agents’ solvability, and (ii) how agents form their financial links and shape the network, depending on their expected payoffs. We deduce the equilibrium state of the network, and we test the effect of a prudential ratio. Fully efficient when agents optimize in the long run, the prudential ratio may also become counterproductive when agents are myopic.

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1 Introduction

1.1 Motivations

At the root of this piece of research, like many authors, we had the idea to see how financial distress percolates through a financial network. The idea of percolation is associated with two phases of the network: one is such that agents are not connected enough, and defaults remain locals. The other one is such that financial links allow defaults to propagate all over the network. The first phase appears at the left of some critical number of links, while the second one appears at the right of it.

For the transmission of defaults, authors have been adding various resistance mechanisms of the agents. However, this percolation phenomenon takes place when the probability to put an edge between any two agents is of type $\frac{z}{n}$ with $z \approx 1$. One contribution of this work is to relax this restrictive assumption and uncover an interesting second regime. In addition, adding random links in a network is also a very specific assumption. But still we believe that the network representation of the financial system is relevant.

In our model, we do not consider purely random networks. We suppose that financial links are built to satisfy conditions on the balance sheets of the agents. We mainly focus on financial intermediaries such as banks, funds, insurance companies. For example, if an agent has a large amount of assets, we suppose that he must also have a large amount of liabilities. We allow the number of financial connections to widely exceed the percolation threshold of the network. Moreover, we consider a large but finite number of agents. We do not investigate into the asymptotic properties of the network. We want to reproduce two distinguishing features of recent financial systems:

- the network has an influence on the financial entities. For example, what is the effect of the defaulting bank on its counterparties, and on the counter-parties of the counter-parties?

- The financial entities build and shape the financial network, for example, by choosing their counter-parties.

Using such a network, we want to capture the high level of financial connections, the possibility of systemic risk and determine the role of the regulator: increasing the capitalization ratio forces the agents to increase the capital, which makes them more resilient to the transmission of financial distresses, but it also reduces the number of financial links in the economy, which may decrease the risk sharing.
1.2 Overview of the model

We consider a network populated by a large (but finite) number $n$ of agents owning the same amount $1$ of capital $C$. We limit the amounts of their assets by a capitalization ratio $\phi$.

We suppose that agents make investments towards other agents. All investments have the same financial size, that we normalize to $1$. An investment lasts one period. When it matures, the investment is supposed to be repaid to the issuer. In this case, the issuer has a net positive payoff $e$. Alternatively, if the investment is not repaid, the issuer of the investment loses the value of his investment.

Financial agents are infinitely lived and optimize the discounted sum of their expected payoffs. At each period, they choose the number of investments they issue, and the destinators of their investments. This determines the structure of the financial network.

Shocks on the network are endogenous, because they correspond to strategically defaulting agents: when some agents are destinators of a large number of investments, and issue less investments, they can choose to default. In this case, they keep the received investment (or liabilities) and they give up their own investments (or assets). Giving up one’s investments somehow corresponds to a fine.

Because of these strategically defaulting agents, some issuers lose their investments, and so might also default, because their liabilities exceed their remaining assets added to their capitalizations. In this case we assume that these agents are liquidated and their assets lose their entire value. Depending on the maximization horizon of agents, short (one period), long (infinite periods) or myopic (intermediary cases), we derive the number of financial links of the network, the number of strategically defaulting and contaminated defaulting agents, and the payoffs of each type of agent. We analyse the role of the regulation on the limitations of the number of investments. We prove theoretically that well-defined solutions to the problem exist. Then we simulate the solution using matlab, and we state our results.

1.3 Implications and results

Short-horizon maximizations lead to more strategically defaulting agents than long-horizon maximizations. This result corroborates the intu-

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$^1$The model does not focus on concentration phenomenons, like TBTF, which would require a small number of bigger agents.

$^2$loosely speaking: a finite number of periods.
The network converges to a dense state when agents optimize in the long-run and sparse state when agents optimize over one period. However, for myopic behaviour, the state of the network of agents depends on their capitalization: for highly capitalized agents, the optimal choice is to create a dense network, which includes a non-negligible systemic risk: all agents are defaulting. Limiting the assets of agents by a prudential ratio decreases the systemic risk but does not completely eliminate it. For low capitalized agents, such a prudential ratio may have counter-productive effects, because it may create a sparse network, and dry up the liquidity.

Section 2 lays the model of the network as a matrix and details how some agents enter strategic default while others do not. In Section 3 we analyse the investment decisions of financial agents as functions of their expected payoffs. Section 4 presents the results of the simulation of the network in short-term optimisation. Section 5 extends the results to all time-horizons, and delivers the policy implications. Appendix B describes the heuristic way of computing simulations of the model.

2 Modelling the network

2.1 Financial connections and network properties

The financial system is a network populated by a large number $n$ of agents or financial intermediaries (traders, investments banks, insurance companies...). To simplify, all agents are supposed to be rational and risk neutral. We assume that agents are homogeneous: they all have the same endowment, also called capitalization. Agents are infinitely lived and time is discrete. In the model, an “investment” is a financial contract.

Financial intermediaries make investments of the same financial size towards other financial intermediaries, at each period, expecting some positive payoffs. This constitutes the financial network. Investments last one period. At the beginning of each period, agents simultaneously choose their targets for investments and the number of their investments. These investments are supposed to be paid back at the end of the period: this is the clearing process. Because investments change at each period, there is a new realization of the financial network at each period. As in Anand, Gai and Marsilli (2012), the agents’ investment choices are not purely random. We assume that agents

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3 The results still hold for risk-averse agents, as the concavity of the utility function would not impact results to a large extent.
know the number of investments planned by the other agents. This means that agents have an accurate estimation of the number of assets of other agents, while they ignore their number of liabilities. This catches the difficulty to evaluate the assets-liabilities ratio of the agents and models counter-parties risks. In other words, agents do not know the exact structure of the network. They choose the number and the destinators of their own investments to maximize their expected payoffs. When an agent makes an investment towards any other agent, he expects to get back his investment at the end of the period, with a net positive yield.

A graph representation of the financial network is possible, as in Allen and Gale (2000) or Gai and Kapadia (2010). Each agent is represented by a dot. Financial flows between agents are represented by oriented edges between the different dots. The directions of the arrows represent the claims on investments, i.e. the directions of the financial flows at the end of each period. An example of financial network is provided on Figure 1. For a matrix representation, see Appendix A.

Agents expect to make profitable investments: there is an arbitrary payoff $e$ such that any refunded investment is supposed to return some gain. By contrast, an investment that is not paid back is definitively lost, and costs one unit (the amount of the investment). From a financial point of view, the parameter $e$ may represent the real interest rate on a financial contract on the interbank market, for example.

At each period $t$, each financial agent $i$ receives some number of
investments also called liabilities\(^4\), \(L^t_i\) and makes a certain number of investments (also called assets), noted \(A^t_i\). These two numbers can be expressed using the matrix of the financial links; see Appendix B.

At the end of the period, any agent has to refund the investments he received, and his investments are also redeemed. We assume that the investments of each financial agent \(A^t_i\) are limited to a fraction of the capitalization\(^5\) \(C\) by a prudential debt to capital ratio \(\phi\) fixed by the regulator, such that:

\[
\phi A^t_i \leq C. \tag{1}
\]

This constraint captures the usual capital requirements. For each agent, the following relation between his assets \(A_i\) and his liabilities \(L_i\) must be satisfied, illustrating the balance sheet:

\[
C + L^t_i \geq A^t_i. \tag{2}
\]

When \(C + L^t_i - A^t_i > 0\), there is some cash that is not invested by agent \(i\).

To summarize, at each period, financial agents make investments given the number of investments of other agents. The allocation of investments is restricted to the networks, such that, for each agent, both conditions (1) and (2) are satisfied.

### 2.2 How agents form their investments

Agents make investments because they expect positive payoffs. Let \(G^t_i(A^t_i)\) be the expected gain of agent \(i\) from his investments \(A^t_i\) at time \(t\), and \(\beta\) the discount factor. The maximization process of each agent can naturally be represented by a Bellman equation. Let \(V(A^t_i, L^t_i)\) be the value function of agent \(i\), depending on the number of his assets \(A^t_i\) and his number of liabilities \(L^t_i\).

#### 2.2.1 Strategic endogenous default

Within the financial network, given agents’ investments, at time \(t\), some agents are net creditors: \(A^t_i > L^t_i\) (they make more investments than they receive) and some other agents are net debtors \(L^t_i > A^t_i\). The net debtors might be interested in strategically defaulting and quitting

\(^4\)An agent has no choice but to accept an investment from any other agent.

\(^5\)Recall that all agents are equally capitalized: \(\forall (i, j), C_i = C_j = C\). In addition, we suppose that capitalization is constant over time.
the network during the period: they will not pay back the investments they received at the beginning of the period. They strategically default if the gain of this action exceeds the discounted sum of their expected payoffs from investments. In this case, they definitively leave the financial network. A strategically defaulting agent leaves the network before the end of the period and therefore also gives up his own investments. The corresponding fine associated to this behaviour is a foreclosure of assets. This form of default allows to endogenize the formation of defaults, while most of the literature concentrates on the effect of an exogenous shock on the financial network. Making a strategic default, because the gain of default exceeds the fine, consisting in giving up one’s investments was introduced in general equilibrium theory with incomplete markets by Dubet, Geanakoplos and Shubik (1989).

Actually, this type of default does not guarantee a particularly high payoff to the strategic defaulting agent, because he gives up his investments. The choice of the settlement mechanism when an agent defaults (keep a fraction of the assets, give up the assets, loose a part of the pledged capitalization) usually relies on the legal context. In our model, we decide that a strategic defaulting agent is giving up his investments, because it is an intermediate way to penalize the default, it does not protect especially the defaulting agents or their victims (the investors). In the introduction, we explained that subprime borrowers were also making strategic defaults because prices of their houses were lower than the amounts of their loans. Long before this crisis, it was known that households could make strategic defaults by filing bankruptcy, as explained by White (1998). The protection of investors and the enforcement law processes influence the choices of the investors and determine the size of capital markets, as proved by Porta, de Silanes, Shleifer and Vishny (1997). As a consequence, the choices of the investors (number of investments) are partially depending on this particular mechanism.

As strategically defaulting agents leave the network without reimbursing their creditors, this impacts the solvency of other creditors who may not be able to honour their obligations and may thus become defaulting by contamination. This also reduces the expected payoff of other agents who still remain solvent. Using the contagion mechanism and the structure of the network, we determine the number of defaulting agents \( D \), depending on the number of strategically defaulting agents can be compared to the “black holes” of other models in the network literature, for example Rotemberg (2009).

\[ \text{such as liquidity excess demand or decrease of the value of the assets.} \]
ing agents $S$.

Strategically defaulting agents are leaving the network, and contaminated defaulting agents collapse, we assume that these two categories of agents are replaced by new agents with the same capitalization $C$ at the beginning of the following period. This allows to keep a constant number of agents over time in the network. This assumption is standard in network models with defaulting agents in a dynamic set-up, see for example in Anand et al. (2012).

At each period, each agent makes a choice, either he stays in the network until next period, or he defaults:

\[ V(A_t, L_t) = \max(\text{strategically default, stay at least one more period})^8, \]

\[ V(A_t, L_t) = \max \left( L_t - A_t, \mathbb{E}_t \left[ G_t(A_t) + \beta V(A_{t+1}, L_{t+1}) \right] \right). \]

### 2.2.2 Contaminated defaulting agents

Any agent expects to get back his investments $A_i$ at the end of the period. Some of them might not be actually paid back as we shall see later. Let $AR_{t+1}^i$ be the investments refunded to agent $i$. If $AR_{t+1}^i + C < L_t^i$, agent $i$ does not have enough cash to refund all his liabilities. In this case, the agent is said to be “contaminated defaulting”. He does not refund any of his creditors, because we assume that the liquidation value of his assets is zero. Like Allen and Gale (2000) and Anand et al. (2012), we use a “zero recovery assumption” when there is a default$^9$.

On the opposite, any agent $i$ such that $AR_{t+1}^i + C > L_t^i$ refunds all his creditors, and is said to be “healthy”. To calculate the exact number of defaulting agents, we study the contamination of the defaults along the paths of investments also called lines of credit. As explained by Watts (2002), paths may have different forms that account for their resistance: trees are more vulnerable than cycles.

The higher the capitalization $C$, the easier it is for an agent to resist default on an investment. From a macroeconomic point of view, the prudential ratio also represents the capacity of the network to limit the proliferation of financial distress along credit chains, as will be proved later on.

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$^8$This Bellman equation can be interpreted as that of the McCall (1970) employment search model.

$^9$This was a major feature of the recent crisis. Relaxing this assumption, for example, refunding a random part of the creditors would generate significantly less defaulting agents.
At the end of the period, we distinguish:

- **$S$** strategically defaulting agents: $L_i^t - A_i^t > G_i^t + \beta E_t \left[ V(A_{i+1}^t, L_{i+1}^t) \right]$;

- $(n - S)$ remaining agents in the network: $L_i^t - A_i^t < G_i^t + \beta E_t \left[ V(A_{i+1}^t, L_{i+1}^t) \right]$,

among which:

- $H$ healthy agents, whose situation is such that $L_i^t \leq C + A R_i^t$: the liabilities $L_i^t$ healthy agents pay at the end of the period do not exceed their own capitalization $C$ added to their refunded investments $A R_i^t$. These agents reimburse their creditors and resist the contamination of the default(s).

- $D = (n - H - S)$ contaminated defaulting agents: the liabilities contaminated defaulting agents have to repay at the end of the period exceed the refunded investments, added to the capitalization. The counter-parties are not paid back.

The estimated number of healthy agents $H$ is therefore a function of the number of agents’ investments, the level of capitalization $C$, the prudential ratio $\phi$ and the number of strategically defaulting agents $S$. The following step of the reasoning is to determine the number of investments agents make each period. With these numbers, we can derive the structure of the network and the number of strategic defaulting $S$ and defaulting agents $D$ and the expected payoff of all agents.

## 3 Optimization

### 3.1 Number of investments

The higher the number of strategically defaulting agents, the higher the number of contaminated defaulting agents in the financial network. As a consequence the more strategically defaulting agents, the lower the expected payoff of agents.

**Proposition 3.1.** There is a unique strategy of investments that minimizes the number of strategically defaulting agents in the financial network: all agents make the same number of investments.

The proof can be understood as the optimal choice of the targets of investment; see Appendix C.

We showed that all agents make the same number of investments. We can determine precisely the number of strategically defaulting,
contaminated defaulting and healthy agents in the economy as a function of the number of investments by agent.

Agent $i$ strategically defaults if and only if $L_i^t - A_i^t > G_i^t + \beta t \mathbb{E}_t \left[ V(A_{i+1}^t, L_{i+1}^t) \right]$. When at least one agent is strategically defaulting in the economy, there might exist contaminated defaulting agents because some investments will not be paid back. Logically, the outcome depends on the level of capitalization of agents. The expected payoff of an agent depends on his total number of investments, and on the financial situation of the counter-parties of his investments. Let $P(H)$ be the probability to reach a healthy, $P(D)$ a contaminated defaulting, or $P(S)$ a strategically defaulting agent with an investment. We consider agent $i$. The expected payoff $G_i(A_i)$ from investments of agent $i$ (risk neutral), who does not strategically default, is derived in the following proposition$^{10}$:

**Proposition 3.2.** When all agents make each $A$ investments, the expected payoff of an agent staying in the network is:

$$G(A) = A ((e + 1)P(H) - 1 + P(S)).$$  

(5)

A proof is provided in Appendix D.

**Proposition 3.3.** The value function $V$ is well defined and unique.

Let $T$ be the operator defined as follows:

$$V(m, L_i^t) = \max (L_i^t - m, \mathbb{E}_t [G_i(m)] + \mathbb{E}_t [\beta V(m, L_{i+1}^t)])$$

$$= T(V(m, L_i^t)).$$  

(6)

$T$ is a contraction. See proof in Appendix D.

### 3.2 Number of strategically defaulting agents

If no agent is strategically defaulting, there is no financial distress, and for each agent $A_i = AR_i$. In this case, the payoff for all agents is $G_{\max} = Ae$. In this particular context, no agent is strategically defaulting except if at time $T$:

$$L_i^T - A > \sum_{t \geq T} \beta^{t-T} G_{\max} = \frac{Ae}{1 - \beta}.$$  

(7)

$^{10}$To simplify notations, we removed the $t$ time index.
This condition is equivalent to: $L_i^T > A(1 + \frac{e^{1-\beta}}{1-\beta})$. Using this condition we determine the number of initially strategically defaulting agents. If some agents are strategically defaulting because condition (7) is fulfilled for some $i$, we call them initially strategically defaulting agents, because they will not pay back their investments, and they will create contaminated defaulting agents. For this reason, by equation (5), we know that $G(A) < G_{max}$. Because $G(A)$ is lower than $G_{max}$, it might happen that some other agents strategically default. To calculate the exact number of strategically defaulting agents, we must evaluate the term $V^* = \mathbb{E}_t \left[ V(A_{i+1}, L_{i+1}) \right]$. By induction, we have:

$$
\mathbb{E}_t \left[ V(A, L_{i+1}) \right] = \mathbb{E}_t \left[ \max \left( L_{i+1}^T - A, G(A) + \beta \mathbb{E}_{t+1} \left[ V(A, L_{i+2}) \right] \right) \right] = \mathbb{E} \left[ \max (L_i - A, G(A) + \beta V^*) \right].
$$

(8)

In the previous equation, we remove the $t$ subscripts, because there is no alea.

**Proposition 3.4.** The term $V^* = \mathbb{E}[V(A, L_{i+1})]$ represents a reservation payoff of staying in the network, and postponing the decision to default strategically to the next period. $V^*$ is well defined, and only depends on $A$.

This proposition is crucial. The value of $V^*$ influences the number of strategically defaulting agents, and therefore the number of contaminated defaulting agents, and the expected payoff of the agents. For this reason $V^*$ “influences its own value”. Appendix E shows that $V^*$ exists and is unique. Otherwise, we would not have a solution to the Bellman equation (4).

### 3.3 Distribution of the liabilities

We consider a network in which all agents are making exactly $A$ investments. To satisfy the balance sheet condition, the agents must receive at least $A - C$ investments. This means that $C$ investments by agents are purely randomly distributed over the network. Let $P(L_i = x + A - C)$ be the probability that agent $i$ receives $x + A - C$ investments. $\forall x \in [0, nC]$:

$$
P(L_i = x + A - C) = \left( \frac{1}{n} \right)^x \left( \frac{n-1}{n} \right)^{nC-x}.
$$

(9)

This distribution allows to calculate the initially strategically defaulting agents of the network. The expectation of the number of initially strategically defaulting agents is:
\[ E[IS] = \sum_{x=0}^{nC} \mathbb{1}_{x-C>G_{\max}} \mathbb{P}(L_i = x + A - C), \]
\[ = \sum_{x=0}^{nC} \mathbb{1}_{x > A + \frac{A}{1-\beta} + C} \left( \frac{1}{n} \right)^x \left( \frac{n - 1}{n} \right)^{nC-x}. \]  

We can also deduce the total number of strategically defaulting agents.
\[ E[S] = \sum_{x=0}^{nC} \mathbb{1}_{x-C>A(e+1)\mathbb{P}(H)-1+\mathbb{P}(S)} \left( \frac{1}{n} \right)^x \left( \frac{n - 1}{n} \right)^{nC-x}. \]  

The probability of reaching a healthy agent with an investment is:
\[ P(H) = P(AR + C > L_i), \]
\[ = \sum_{L_x=A-C}^{A+(n-1)C} P(AR + C > L_x)\mathbb{P}(L_i = L_x), \]
\[ = \sum_{L_x=A-C}^{A+(n-1)C} \sum_{h=0}^{n-1} A\mathbb{P}(H = h)\mathbb{1}_{Ah+C>L_x} \mathbb{P}(L_i = L_x). \]

In practice, we construct a sequence, converging to the values of \( S^* \), \( G(S^*) \) and \( V^* \): we start from \( S_1 \) the initially strategically defaulting agents of the network, those such that relation (7) is verified. Then we calculate over the network the spread of financial distress, i.e. the number of contaminated defaulting \( D_1 \) and healthy \( H_1 \) agents. This gives \( G_1 \) by equation (5). Using (29), we calculate \( V_1 \). Using \( V_1 \) and the Bellman equation (4), we determine the new number \( S_2 \). Then we calculate, \( D_2 \) and \( H_2 \), and \( G_2 \). Using (29) we calculate \( S_3 \), and so on. This sequence of \( (S, D, H, G) \) converges to the solution.

We proved that agents make the same number of investments at each period. We know how to calculate the number of strategically defaulting, contaminated defaulting and healthy agents, as well as the expected payoff function of the agents, depending on the number of investments by agent. Agents optimize their expected payoff by choosing the number of investments they make. The regulator limits the number of investments by the prudential ratio to maximize the number of healthy agents. In the model, because investments’ amounts are normalized to 1, the number of investments determines the connectivity of the network. Increasing the capitalization ratio decreases the connectivity of the network. \( \phi \) does not a priori limit the
amount of investments, however, there exists an internal threshold within any financial intermediary, which limits the financial position of the intermediary on any financial contract with one counter-party. As a consequence, $\phi$ does not only limit the number but also the volume of the transactions of agents in the network.

In the sequel, we study how the agents’ problem (optimize the discounted sum of profits), and the regulator’s problem (maximization of the number of healthy agents) are compatible. To simplify the problem, we analyse first the short term case, where agents play only one period. Then we see how the infinite periods game changes the results. Intermediary cases, when agents are myopic and base their expectations on a limited number of periods, will be presented briefly in the last section.

4 Particular case: simulation of the one-period network

A description of the method used to produce simulations is provided in Appendix F.

4.1 Initially strategically defaulting agents

Technically, the one-period network is equivalent to consider $V^* = 0$ in the Bellman equation (4). We remove the $t$ subscripts and the recursive equation becomes:

$$V(A, L_i) = \max (L_i - A, G(A)).$$

The payoff equation remains the same:

$$G(A) = A ((e + 1)p(H) - 1 + p(S)).$$

The equation determining the number of initially strategically defaulting agents slightly changes and becomes:

$$L_i - A > G_{\max} = Ae.$$  

Figure 2 is plotted using the following parameter values: $n = 400$, $\phi = 7\%$, and $C = 10$ (arbitrary units). The maximal number of investments by agent is determined by the value of the prudential ratio: $A_{\max} = \frac{C}{\phi} = 142$. We simulate over a large number of networks.
Figure 2: Number of initially strategically defaulting agents w.r.t. the number of investments by agent.

(≈ 2000) the average number of initially strategically defaulting agents of the network depending on the payoff by investment from $e = 2.5\%$ to $e = 10\%$ on Figure 2.

All the curves representing the number of initially strategically defaulting agents have the same shape:

- After an initial increase, each curve is decreasing with respect to the number of investments by agents,
- the curves are not smooth: there are regular thresholds of decreases.

These observations can be easily explained. For each value of $e$, the regularity of the “sudden decreases” of the curve corresponds to values of $A$ such that $Ae$ is an integer. Indeed, the number of liabilities minus the number of assets is obviously an integer, therefore the comparison threshold between the net gain from strategically defaulting and the gain from investments does not change when $Ae$ keeps the same integer part. The periodicity in terms of investments is calculated the following way: $\pi_e = \frac{1}{e}$. For the first $C$ investments, there is
no constraint about the allocation of investments because each agent satisfies $A_i \leq C$: the allocation is purely random. When the number of investments by agent increases and remains smaller than $C$, the probability to get more investments than the average increases: the number of strategically defaulting agents increases – as long as $Ae$ keeps the same integer part. Once the average number of investments by agent is larger than $C$, each agent making $A$ investments must receive at least $(A - C)$ investments in order to have the balance sheet condition (2) satisfied. This situation is equivalent to distribute randomly $C$ investments made by agent over the whole network and $(A - C)$ investments exactly received and made by each agent. The probability to get $x$ more investments than $A$ is therefore constant with respect to $A$, whatever the value of $A > C$. The global decrease of the curve is due to the increasing gains due to investments: $Ae$.

4.2 Viability of the short term network

Using the number of initially strategically defaulting agents, we calculate the number of defaults in the network. These defaults reduce the expected gains of the other agents, who therefore have an incentive to strategically default as well. We determine the stationary equilibrium number of all strategically defaulting agents in the network. For Figure 3, we adopt $e = 5\%$. With the exact number of strategically defaulting agents, we also know the number of contaminated defaulting agents and the number of healthy agents in the network.

To determine the exact number of investments agents make, we plot the expected payoff by agent depending on the number of investments on Figure 4, for different values of the parameter $e$.

On Figure 4, we observe the expected payoff of agents as a function of the number of investments by agent. For $e = 10\%$, the expected payoff is strictly increasing with respect to the number of investments by agents. As a consequence, the unique equilibrium choice of agents is to make exactly $A_{\text{max}} = \frac{C}{e}$ investments.

If we consider the curves for which $e = 2.5\%$ or $e = 5\%$, there exists a zone such that the expected payoff is lower than or equal to zero. Agents do not enter the network if the expected payoff is negative. For 40 investments by agents, the expected payoff is strictly positive and reaches its maximal value. Agents will therefore coordinate and make exactly 40 investments each. This situation illustrates some features

\footnote{In terms of graph theory, we can say that the in-degree of each node is larger than $(A - C)$ and the out degree is exactly $A$.}
of the recent crisis. Agents make investments, but it is optimal not to invest as much as possible, because it generates too big distresses, and negative payoffs. During the interbank crisis in 2008, banks refused to lend (invest in the model), fearing that their counter-parties may be in financial distress, which generated a liquidity squeeze.

For the intermediary value $e = 7.5\%$ there is a “tricky” situation, the expected payoff is positive for $A \leq 55$ to $A \geq 75$ investments by agents. The choice of the agents in terms of expected payoff depends on the value of the prudential ratio. If $A_{\text{max}} \leq 85$ which is equivalent to $\phi \leq 12\%$, the agents choose to make $A_{\text{max}}$ investments. On the contrary if $\phi > 12\%$ agents choose to make $A = 55$ investments.

We state the conclusions of the one-period networks:

- for high values of the return on investments ($e > 8\%$), agents make the highest number of investments, limited by the prudential ratio $\phi$;
- for low values of the return on investments, agents choose to
make a lower number of investments than the theoretical limit fixed by the regulator. This may represent liquidity problems;

- for very low values of the return on investment, risk-neutral agents do not play in the network. Risk-seeking agents may invest even if the expected payoff is negative because there is still a positive probability to be one strategic defaulting agent, whose net gain is strictly positive;

- agents enter the network and make investments even if there is a net positive number of strategically defaulting or contaminated defaulting agents.

From a social welfare perspective, the regulator seeks to minimize the number of strategically defaulting and contaminated defaulting agents, and maximize the number of healthy agents. On the previous example, using Figure 3, we remark that this objective may correspond to the maximization of the expected payoff of agents. However, these two objectives may be incompatible.
For some parameter values, the maximization of the expected payoff and the maximization of the number of healthy agents are conflicting, see for example on Figure 5, $\phi = 10\%$ and $\epsilon = 8\%$:
The expected payoff is on average increasing with the number of investments by agent while the number of contaminated defaulting agents reaches its maximum for some bounded value. Agents want to make $A_{\text{max}} = \frac{C}{\phi}$ investments while the regulator would prefer the agents to make either more than 80 or less than 50 investments. The only solution for the regulator is to choose either $\phi < 12.5\%$ or $\phi > 20\%$.

To achieve his goals, the regulator is interested in increasing the value of the prudential ratio $\phi$ to limit the number of investments by agent, but this is likely to reduce the expected gains of agents.

What is worse between strategically defaulting and contaminated defaulting agents? The strategically defaulting agents make the choice to default, initiate the defaults, and are responsible for the problems, and the contaminated defaulting agents are conse-
quences of the strategically defaulting agents. On the one hand the regulator tries to banish strategically defaulting agents even if there are few contaminated defaulting agents, on the other hand the regulator tries to limit the proliferation of financial distresses coming from a few number of strategically defaulting agents: the regulator arbitrates (trade-off). Contaminated defaulting agents are more costly than strategically defaulting agents, because they do not refund any of their creditors, while strategically defaulting agents give up their own investments.

5 Generalization: Infinite time model

The one-period network yields a very large number of defaulting agents, which is quite unrealistic. We therefore concentrate on the more realistic case of an infinite time network. To determine the number of strategically defaulting agents, we introduce the $V^* \neq 0$ parameter in the Bellman equation (4). Even if there is a well-defined expression for $V^*$, all the probabilities and also the contamination are difficult to evaluate. $V^*$ is estimated by a recursive method.

5.1 Determination of the postponing threshold $V^*$

As we already know, from time $T$, $V^* > \sum_{t>T} \beta^{t-1} G_t^i(A^i)$. Starting from $V^* = 0$ we deduce $A^*$ such that $G^i(A^*)$ is maximal. Then we replace $V^*$ by $\frac{1}{1-\beta} G^i(A^*)$ and we compute again the new expected numbers of strategically defaulting, contaminated defaulting and healthy agents, and we also deduce the expected gain function $G^i(A^i)$. Then we determine the new $A^*$ which maximizes $G^i$. We replace $V^*$ by $\frac{1}{1-\beta} G^i(A^*)$, until there is convergence. We assume that this lower bound estimation of $V^*$ is close to its real value as long as the probability $P(S)$ to reach a strategically defaulting agent with an investment remains very low.

To get the best accurate value of $V^*$, there is an important discussion about parameters $e$ and $\beta$. The discount factor $\beta$ is not supposed to change over the period and is common to all financial entities. It is related to the long-term risk free rate $r_\infty$. On the opposite, $e$ is the return on a one-period repaid investment. It is related to the short-term interest rate.

Suppose we know $G_{\max}$. Then $\sum_{t>0} \beta^t G_{\max} = \frac{\beta}{1-\beta} G_{\max}$. If the rate of return of the capital is constant, then $\beta = \frac{1}{1+e}$. If also $G_{\max}$ is proportional to $e$: $G_{\max} = Ae$, the network is indifferent to the value of
e. For high values of $e$, when almost no one is strategically defaulting, this is true. On the opposite case, for low values of the return on investment, $G_{\text{max}} \neq Ae$, because there are strategically defaulting and contaminated defaulting agents. Introducing a net shift $V^*$ in the maximization equation will modify the behaviour of agents: there will be less strategically defaulting agents.

We assume that the long term interest rate is $r_\infty = 2.5\%^{12}$. In this case $\frac{2}{1-\beta} = 49$. To evaluate $V^*$, we shall consider the corresponding gain. For $e = 2.5\%$ the maximal expected payoff of the short-term network is $G \approx 0.38$. A lower bound for $V^*$ is $V^* = 18$. As shown on Figure 7, taking $V^* \geq 10$ is large enough to completely eliminate strategically defaulting agents. As a consequence, there are no contaminated defaulting agents, all financial agents are healthy.

We show the results of simulations, obtained for different values of $V^*$, from the one period model $V^* = 0$ to $V^* = 10$. This represents the state of the network depending on the weight financial agents put on the future payoffs.

As we observe on Figure 6, as soon as $V^* > 5$, the number of initially strategically defaulting agents is highly reduced compared to the one-period case. When $V^* > 10$, there are no initially strategically defaulting agents anymore.

On Figure 7, we present the total number of strategically defaulting agents; $S$ remains very close to the number of initially strategically defaulting agents. This is easy to understand: since $V^* \gg G_{\text{max}}$, the effect of the initially strategically defaulting agents on $G_{\text{max}}$ is negligible compared to the value of $V^*$. Then we derive the number of healthy agents.

---

12We choose to adopt realistic parameters values like Nier, Yang, Yorulmazer and Alentorn (2007), even if some other range of parameters may lead to more impressive results. For the number of banks, we can take into account the work of Bech and Atalay (2010).
5.2 Global results

In this section, all the simulations correspond to $e = 2.5\%$. On Figure 8, the number of healthy agents is quite high, as soon as $V^* > 2.5$. To find the number of investments by agents, we show the estimated expected payoff by agent on Figure 9.

Given the simulations on the expected payoffs depending on the value of $V^*$, we remark that the payoff is strictly increasing if $V^* > 5$. This proves that when $n = 400$ and $C = 10$ in the infinite time horizon, the financial network reaches by itself a stable and healthy situation. For $V^* = 5$ the expected payoff is on average increasing, but decreases when the number of investments by agents exceed 120. This curious behaviour is the representation of the systemic risk. To understand this key issue, we zoom in Figure 10 on the number of healthy agents for $V^* = 5$.

The curve of healthy agents is obtained by taking the average number of healthy agents over a large number of simulations. The curve is quite smooth for $A < 100$. Indeed, the difference between the total

Figure 6: Number of initially strategically defaulting agents w.r.t. the number of investments
number of agents $n = 400$ and the number of healthy agents corresponds to the number of strategically defaulting agents, as drawn on Figure 7. However for $A > 100$ there are a few strategic defaulting agents: $S < 4$. As a consequence, there is a net positive number of contaminated defaulting agents. Precisely, among the different simulations, a large number of them have a high number of healthy agents: $H > 390$ and some others have 0 healthy agents, and $D > 390$ contaminated defaulting agents. Taking the average over these simulations makes the average curve very noisy, even with $\approx 1500$ simulations. This proves that systemic risk exists, because a few number of simulations lead to the whole default of the economy. Using the simulations, we can deduce the frequency of systemic defaults: in the range 100 to 115 investments by agent, there is on average 1 situation of systemic default on 1000 simulations, which represents a risk of 0.1%. From 115 to 130 investments by agent, there is a systemic risk of 0.2%. Above 130 investments by agent, there is a systemic risk of 0.5%. In this case, fixing the prudential ratio to $\phi = 10\%$ avoids to exceed $A = 100$ investments by agents, and prevents systemic risk, without reducing by
Figure 8: Number of healthy agents w.r.t. the number of investments by agent

a large amount the expected payoff of agents. The prudential ratio eliminates the systemic risk.
• Long-run networks have a lot less strategically defaulting agents than short-term networks.

• When agents are risk neutral and optimize over their infinite lifetime, when the discount factor and the return on investment can be linked: $\beta = \frac{1}{1+\tau}$, the network reaches a state containing only healthy agents.

• The prudential ratio $\phi$ limits the number of investments by agents. If agents adopt a myopic behaviour, some cases of default appear for a high level of investments by agent, this represent the systemic risk. Unlike conclusion of most other networks models, more complete networks have a higher systemic risk than less dense networks. Setting properly the value of the prudential ratio $\phi$ avoids systemic risk.

If agents attach less importance to their future payoffs, we say that they adopt a myopic behaviour. This myopic behaviour may represent
agents who expect to achieve their payoffs in finite time, or agents who fear the long-term equilibrium of the network (risk of crisis). This is equivalent to reduce $V^*$. Precisely, we define a myopic agent as an agent who only takes into account the gains of the current period and the 3 following ones. We determine in the next section the performance of the prudential ratio on the different situations: short-term, myopic, long-run.

5.3 Robustness of the ratio

In the two previous sections, we studied the case of a financial network containing $n = 400$ agents, all of them owning a capital of $C = 10$ units. We raise the question whether the results and the policy implication change with respect to these variables. By simulation, we prove that the results hold when $n$ changes, as soon as $n$ remains large enough. When the capitalization of agents changes, the results evolve. When agents have a lower capitalization, the number of assets is close to the number of liabilities: agents make $A$ investments and must have at least $A - C$ liabilities. As a consequence we could expect less strate-
gically defaulting agents. However, with less capital and the same prudential ratio, the maximum number of investments is lower. Table 1 determines which effect dominates. We fix $e = 2.5\%$ and $\beta = 0.98$, we let $C$ vary.

On Table 1, we compare the results obtained with different values of the capitalization, for two choices of the prudential ratio 5\% and 10\%, and for three types of behaviours of agents: one-period optimization, infinite lifetime and myopic behaviour. In the table, we show the results of simulations for different capitalizations: from $C = 4$ to $C = 10$ (arbitrary units). We determine the maximal number of assets $A_{\text{max}}$, which only depends on the capitalization $C$ and the prudential ratio $\phi$: $A_{\text{max}} = \frac{C}{2}$. We deduce the theoretical maximal gain $A_{\text{max}}e$. Then, by induction, we determine the three terms $V^*$, $G^*$ and $A^*$, which represent the optimal choices of the agents.

To determine whether $\phi$ is effective, we distinguish three cases:

- $A^* = A_{\text{max}}$, $\phi$ is fully effective because agents make the maximum number of investments. In this case, the prudential ratio prevents to reach a too connected state, which could present a systemic risk, especially when agents are myopic.

- $A^* \ll A_{\text{max}}$, $\phi$ is useless, agents make a small number of investments. There are liquidity problems. Agents could make more investments, but they do not make them, because they are afraid of contamination. The network would be in bad state (systemic failure) for a high number of investments by agents. This case would be emphasized by risk-averse agents.

- $A^* \approx < A_{\text{max}}$, $\phi$ should be a little reduced to avoid risk of defaults.

Let us focus on a particular case: for $n = 400$, $C = 5$ and myopic behaviour, corresponding to the two red lines of Table 1. When $\phi = 5\%$, agents restrict their investments to $85 < A_{\text{max}} = 100$ investments, to reach the optimal payoff. The equilibrium values of the model are $G^* = 2.1$ and $V^* = 6.3$. This value also minimizes the number of contaminated defaulting and strategically defaulting agents. For any value of $A > 85$ there is a net positive probability of systemic risk, as explained in the previous section. For this reason, the regulator may decide to reduce the maximal number of investments of the agents. By decreasing the maximal number of investments: from 100 when $\phi = 5\%$ to 50 when $\phi = 10\%$, the regulator also decreases the expectations of the maximal gains of agents: from $G_{\text{max}} = 2.5$ to $G_{\text{max}} = 1.25$. This also decreases the maximal value of $V^*$: $V^* \leq 3.75$. This situation
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Table 1: State of the network depending on the capitalization ($e = 2.5, \beta = 0.98$)

creates more strategically defaulting agents. To avoid the risk of contagion, agents decide to make less investments: $A^* = 9$ than the maximal number allowed: $A_{max} = 50$. This situation illustrates that, when the regulator strengthens the prudential ratio to avoid the systemic risk, and agents are myopic, liquidity may dry up. Mechanically, if the capitalization of agents is decreasing, due to an external shock, this could produce the same effects; by reducing the anticipations of the expected payoffs of the agents.

- When agents optimize over their infinite lifetime, even with low capitalized agents, the use of a prudential ratio to avoid major
defaults is necessary and successful.

- When agents optimize over the current period only, the use of a prudential ratio is useless, it cannot reduce the number of strategically defaulting agents. Agents make less investments than they could. Defaults do not propagate.

- When agents are myopic\textsuperscript{13}, the use of a prudential ratio depends on the capitalization of agents in the network:
  - necessary and successful for highly capitalized agents, prevents systemic risk,
  - useless and harmless for weakly capitalized agents, this is the equivalent of one-period optimization,
  - necessary but counter-productive for intermediately capitalized agents: prevents systemic risk, but generates liquidity problems.

5.4 Conclusion

The model of network developed in this piece research has several common implications with the recent literature: the risk of default is less important for very low and very high connected networks. Nevertheless, we must handle with care definitions of low and high connectivity, and incomplete, sparse or dense networks, because these notions vary a lot depending on the network context: a complete network in percolation models (Anand \textit{et al.}, 2012) would be an incomplete network in contagion theory (Allen and Gale, 2000), Figure ???. Our range of connectivity starts with the lowest percolation threshold (1 investment starting from each agent, corresponding to the lowest bound of Watt’s global cascade window Figure ??) and reaches high connected networks ($\frac{C}{\phi} >> 1$ investments by agent), which remain still less connected than the complete network (Figure ??) of contagion theory... Apart from that, our threshold of systemic risk (a large number of investments by agent) does not correspond to the usual percolation threshold (a few investments by agents). Indeed, the threshold of systemic risk mainly depends on the capitalization and the expectations of agents.

\textsuperscript{13}For risk-averse agents, depending on the concavity of the utility function, $V^*$ would be reduced, and this would accentuate the myopic behaviour of agents
Systemic risk exists when agents are myopic, and is increasing with respect to the connectivity. The model has new features, such as the endogenous shocks, represented by the strategically defaulting agents. The model describes in a simple way the effect of agents’ choices on the network and the effects of the network structure on the financial health of agents. The model also takes into account the horizon perspective to determine the optimal behaviour of agents. The model derives policy implications about the use of a capitalization ratio. To conclude, capitalization ratios must be wisely employed, especially when confidence on financial markets is threatened, and when agents tend to adopt myopic behaviours, because prudential ratio, in these situations, may barely trade liquidity crisis for a systemic risk.

Appendix

A Matrix representation of the network

As all the outgoing financial flows and incoming financial flows of any dot have the same standard financial size\(^{14}\), all financial flows can be represented by a \((n \times n)\) matrix \(M\) between agents. For example, to model an investment from agent \(i\) to agent \(j\), we add an oriented edge on the graph from agent \(j\) to agent \(i\), and we add in the matrix the number 1 on line \(i\) and column \(j\): \(M_{ij} = 1\). Any investment from \(j\) to \(i\) is a debt from \(i\) to \(j\), so the matrix representing the links between agents is anti-symmetric \(M_{ji} = -1\). We suppose that only one investment is possible between any two agents, so that multiple and reciprocal investment do not exist. Because investments are reallocated at each period, the matrix also changes. At time \(t\), the matrix is denoted by \(M^t\). For example, the matrix corresponding to the previous graph on Figure 1 is:

\[
M = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
1 & 1 & 0 & 1 & -1 \\
0 & -1 & -1 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 \\
\end{pmatrix}.
\] (15)

\(^{14}\)We normalize the values of all financial flows to 1.
B Expressions of assets and liabilities

Assets and liabilities can be expressed using the matrix of financial links:

\[ A_t^i = \sum_j (M^i_j)^+, \]
\[ L_t^i = -\sum_j (M^i_j)^-, \]

where \( + = \max(0, \cdot) \) and \( - = \max(0, -\cdot) \).

C Proof of Proposition 3.1

\textit{Proof.} For clarity, we remove the \( t \) time index. Agents have the same capitalization \( C \) and are limited to \( A_{\text{max}} = \left[ \frac{C}{\bar{A}} \right] \) investments. Each agent therefore makes between 0 and \( A_{\text{max}} \) investments. Because no reciprocal investment is possible, this leads to \( A_{\text{max}} \leq \frac{n}{2} \). Let \( \bar{A} \) be the average number of investments made by an agent in the network. \( A_i \) is the number of investments made by agent \( i \). We order the agents owing to their number of investments. Let agent \( 0 \) be the “lowest” investor, i.e. the one making the lowest number of investments. On the opposite, agent \( n \) is the “largest” investor.

- Suppose that agent \( 0 \) is making exactly 0 investment. His expected payoff from investing is thus 0. If he receives a least one investment, strategically defaulting guarantees a strictly positive payoff. As a consequence, no creditor will be paid back at the end of the period. Obviously, no one will make an investment towards this agent. Agent making 0 investment is not a good target for investment, the agent is said to be not “attractive”. Similarly, if agent \( 0 \) were to be a low investor: \( 0 < A_0 \ll \bar{A} \), any other agent of the network would fear this agent as a potential strategical defaulter. As a consequence, he will not receive any investment either.

- We can suppose \( 0 < \bar{A} \leq A_n \leq A_{\text{max}} \). If \( A_n = \bar{A} \), all agents make the same number of investments. Any other agent of the network may consider to invest towards agent \( n \) because he expects agent \( n \) not to be – a priori – a strategically defaulting agent. If all agents of the network act this way, agent \( n \) will receive approximately \( (n - 1) \) investments, and he is likely to become a strategically defaulting agent because \( A_{\text{max}} \ll n - 1 \). Thus investing towards this agent is not a good strategy.
Both the lowest and the largest investors are not attractive. Investors restrict their choices to the $(n - 2)$ remaining agents of the network. We consider the new set of agents, from agent 1 to agent $n - 1$, and we apply the same reasoning, until the remaining agents are making exactly the same number of investments.

To decrease the overall number of strategically defaulting agents, agents are willing to distribute evenly the investments among themselves. Any agent which is not receiving investments because he is not attractive (too many or too few investments) forces the other agents to make their investments towards other more attractive agents. This situation creates some attractive agents receiving too many investments: $L_i >> A$ and possibly makes them strategically default. To minimize the number of strategically defaulting agents, the best way is to maximize the number of attractive agents.

To conclude, all agents have to make the same number of investments. From there $A_t$ represents the common number of investments by agents at time $t$.

\[ G(A_t) = E \left[ \sum_{j \in H} e^{I_{i \rightarrow j}} - \sum_{j \in (D \cup S)} 1^{I_{i \rightarrow j}} + \sum_{j \in S} 1^{I_{j \rightarrow i}} \right]. \]  

(18)

The last term of the previous equation $\mathbb{P}((j \rightarrow i) \cap (j \in S))$ corresponds to the probability that agent $j$ is a strategically defaulting agent, and is investing towards agent $i$. Before the end of the period, he leaves the network and gives up his own investment to agent $i$.

\[ G(A_t) = A_t \left[ e^{\mathbb{P}(H)} - \mathbb{P}(D) - \mathbb{P}(S) \right] + \sum_{j \neq i} \mathbb{P}((j \rightarrow i) \cap (j \in S)]. \]  

(19)

By definition, $\mathbb{P}(D) = 1 - \mathbb{P}(H) - \mathbb{P}(S)$, there are $(n - 1)$ directions for each investment, this leads to $\mathbb{P}(j \rightarrow i) = \frac{A_i}{n-1} = \frac{A_t}{n-1} = \frac{A}{n}$, because all agents are making exactly $A$ investments.

Section Proof of Proposition 3.3

Proof. For simplicity, we suppose that we already know the common number – see Proposition 3.1 – $m$ of investments made by agents at each period. We follow Ljungqvist and Sargent (2004) to solve for the Bellman equation of a McCall type of problem. Let $L_t$ be the set of...
incoming links at time $t$. $L_t = L$ is independent of time as soon as the number of investments $A_t = m$ is constant over time. Equation (4) can be written as:

$$V(m, L_t^n) = \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) + \beta V(m, L_t^{n+1}) \right] \right).$$ \hspace{1cm} (20)

The expectation is linear, which gives:

$$V(m, L_t^n) = \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V(m, L_t^{n+1}) \right] \right).$$ \hspace{1cm} (21)

Let $T$ be the operator defined as follows:

$$V(m, L_t^n) = \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V(m, L_t^{n+1}) \right] \right) = T(V(m, L_t^n)).$$ \hspace{1cm} (22)

Let $V_1$ and $V_2$ be two functions, such that for any $0 < L_t^n < N, V_1(m, L_t^n) < V_2(m, L_t^n)$. Then,

$$T(V_1)(m, L_t^n) = \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V_1(m, L_t^{n+1}) \right] \right),$$

$$\leq \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V_2(m, L_t^{n+1}) \right] \right),$$

$$\leq T(V_2)(m, L_t^n).$$ \hspace{1cm} (23)

This proves that $T$ is monotonic. We check that the operator also satisfies the discounting property. Let $\alpha$ be a positive constant.

$$T(V_1 + \alpha)(m, L_t^n) = \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta (V_1 + \alpha)(m, L_t^{n+1}) \right] \right),$$

$$= \max \left( L_t^n - m, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V_1(m, L_t^{n+1}) \right] + \beta \alpha \right),$$

$$\leq \max \left( L_t^n - m + \beta \alpha, \mathbb{E}_t \left[ G_t(m) \right] + \mathbb{E}_t \left[ \beta V_1(m, L_t^{n+1}) \right] + \beta \alpha \right),$$

$$\leq T(V_1 + \alpha)(m, L_t^n) + \beta \alpha.$$ \hspace{1cm} (24)

By Blackwell’s theorem, we know that $T$ is a contraction with modulus $\beta$ on the complete set of functions on $L$ with the sup norm. As a consequence, there is a unique fixed point of $T$. This is the unique value function to the Bellman equation. \hfill \Box
E Proof of Proposition 3.4

Proof. We know that \( V^* \geq 0 \), (0 is the case of no investment or risk loving agents) and \( V^* \leq G_{\max} \sum_{t \geq 0} \beta^t \). Let us write \( V^* = \mathbb{E} \left[ V(A, L_t^{t+1}) \right] \).

The previous equation (8) becomes:

\[
V^* = \mathbb{E} \left[ \max \left( L_t^{t+1} - A, G(A) + \beta V^* \right) \right],
\]

\[
= \mathbb{E} \left[ (L_t^{t+1} - A) I_{L_t^{t+1} - A - G(A) \geq \beta V^*} \right] + \mathbb{E} \left[ (G(A) + \beta V^*) I_{L_t^{t+1} - A - G(A) < \beta V^*} \right]
\]

\[
= \sum_{k=A+G(A)+\beta V^*}^{n-1} k \mathbb{P}(L_t^{t+1} = k) + \sum_{k<A+G(A)+\beta V^*} (G(A) + \beta V^*) \mathbb{P}(L_t^{t+1} = k),
\]

\[
= \sum_{k=A+G(A)+\beta V^*}^{n-1} k \mathbb{P}(L_t^{t+1} = k) + (G(A) + \beta V^*) \sum_{k<A+G(A)+\beta V^*} \mathbb{P}(L_t^{t+1} = k).
\]

No agent would invest if the expected payoff was not positive. The value function is also positive. This explains the second equality.

Agents such that \( L_t^{t+1} - A > G(A) + \beta V^* \) are exactly the strategically defaulting agents, while others remain in the economy. There are exactly \( n - S^* \) remaining agents in the network, where \( S^* \) is the equilibrium value of the number of strategically defaulting agents. We can deduce that

\[
\sum_{k<p+G(p)+\beta V^*} \mathbb{P}(L_t^{t+1} = k) = \frac{n-S^*}{n},
\]

and

\[
\sum_{k \geq p+G(p)+\beta V^*} \mathbb{P}(L_t^{t+1} = k) = \frac{S^*}{n}.
\]

Since all the terms \( \mathbb{P}(L_t^{t+1} = k) \) can be calculated, at least numerically, we can also calculate the values of the following function, which represents the cumulative distribution function of the liabilities:

\[
L(x) = \sum_{k=x}^{n-1} \mathbb{P}(L_t^{t+1} = k).
\]

From the last function, we deduce \( x^* \) such that \( L(x^*) = \frac{S^*}{n} \). This gives the value of \( x^* = A + G(A) + \beta V^* \). We can determine \( G^*(A) = G(A) \) as a function of \( S^* \), because using \( S^* \), we can calculate \( D^* \) and deduce \( G^* \). The global equation (25) on \( V^* \) gives us the theoretical relation between \( S^* \) and \( V^* \):

\[
V^* = \left( \sum_{k=L^{-1}(\frac{S^*}{n})}^{n-1} k \mathbb{P}(L_t^{t+1} = k) \right) + \frac{n-S^*}{n} (G^*(A) + \beta V^*).
\]
For each value of $S^*$, we can calculate a corresponding value of $V^* = V^*(S^*)$. We derive the term $G^*(A) + \beta V^*$. Given the distribution of $L^*_t$, we can determine the expectation $\mathbb{E}$ of the number of agents such that $L^*_t - A > G^*(A) + \beta V^*$. When $S^* = \mathbb{E}$ the solution is obtained. This proves that there exists a well-defined solution. Given $A$, there is a unique $(S^*, D^*, H^*, G^*(A))$. 

\section*{F About the computing}

To obtain the simulations, we have been programming on Matlab. After choosing an arbitrary number of agents $n$, we had to fill the $M$ matrix corresponding to the financial network, such that, for any number of asset by agent, conditions (1) and condition (2) are satisfied. For a number $A$ of investment by agent, we construct recursively the matrix $M$. We add one investment for each agent, then 2 investments for each agent... until $A$ investments by agent. But the investments are not purely random to satisfy (2). In the algorithm, when adding a new investment for all the agents, this requires to create a list of the empty elements of the $M$ matrix, to look for the agents which need an investment to satisfy (2). This step was not obvious because it also needs to look for the agents which can make an investment, and which have not already invest towards this agent. To verify that the matrix is properly filled, we sum the lines and the columns of the positive/negative part of the matrix.

Once the matrix $M$ is filled, we can find the agents which initially strategically default, those such that their net number of investments (received minus issued) exceed the discounted sum of maximal payoffs. Then we calculate using the matrix the effect of these initially strategically defaulting agents, this gives some contaminated defaulting agents. With these two values of strategically defaulting and contaminated defaulting agents, we calculate the new discounted sum of maximal payoffs, and the new number of strategically defaulting agents, and so on. This is possible using a large number of loops.

There is another problem, it is possible to calculate the individual payoffs of all the agents of the period, but difficult to estimate correctly the value $V^*$ of the expected discounted sum of incoming payoffs. This required to try (guess and verify method) a large number of values of $V^*$ and select the one which corresponds to the average payoff for a large number of networks. The simulation of the one period network is a lot easier, because $V^* = 0$, as explained in section 2.4. A simulation of one network of 400 agents with the capitalization $C = 10$ and $\phi = 10\%$
takes about 10 minutes on a modern 2 cores 4 threads processor in parallel computing. To get smooth curves, we must simulate at least 2000 networks, which lasts for 340 hours, about 14 days nonstop. In addition, we had to test a lot of different parameters’ values.

References


