Changes in ambiguity -
Definition, measures and application

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Abstract

This paper investigates the notion of changes in ambiguity over loss probabilities. In particular, we suggest conditions under which an individual always considers one situation to be more ambiguous than another. Changes in ambiguity over loss probabilities are expressed through a specific case of stochastic dominance of order \( n \) as developed by Ekern (1980), and can be interpreted in terms of harms disaggregation over probabilities. We also present two measures to capture changes in ambiguity over probabilities, a non-monetary measure, the ambiguity welfare premium and a monetary measure, the ambiguity premium. To illustrate our results, we examine a problem of optimal resources allocation in the presence of more ambiguity.

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1 Introduction

Since the seminal experiments of Ellsberg (1961), it is well recognized that individuals are averse to ambiguity over probabilities. Ellsberg showed that individuals usually prefer gambles with known rather than unknown probabilities, that is, they are ambiguity averse. Many other experiments have confirmed Ellsberg’s work since then (e.g. Chow and Sarin, 2001; Halevy, 2007), and several models of decision have been proposed to integrate ambiguity preferences in the face of risky situations (e.g. Gilboa and Schmeidler, 1989; Epstein and Schmeidler, 2003; Klibanoff, Marinacci and Mukerji, 2005).

A recent stream of literature has addressed the effect of ambiguity aversion on economic decisions, whether these are insurance decisions (Alary et al., 2013), prevention decisions (Snow, 2011), portfolio decisions (Gollier, 2011), and decisions over the value of statistical life (Treich, 2010). Ambiguity aversion is defined as a preference for non-ambiguous situations over ambiguous situations, i.e. it makes it possible to compare a situation of ambiguity over probabilities to a situation of no ambiguity. However, a more general question is to wonder how an individual compares two situations of ambiguity over probabilities. More precisely, when can we say that one situation is more ambiguous than another? The aim of this paper is to offer a response to this question.

We investigate the notion of changes in ambiguity using the recent theory of ambiguity axiomatized by Klibanoff, Marinacci and Mukerji (2005) (hereafter KMM) that encompasses most common theories of ambiguity. Their approach separates ambiguity preferences from risk preferences. It also introduces a simple way to define ambiguity aversion which is captured through the idea of aversion to any mean-preserving spread in the space of probabilities. This comes from the fact that the introduction of ambiguity constitutes a mean-preserving spread in the space of probabilities.

Consider a probability of loss. In the absence of ambiguity, the decision-maker knows the value of this probability, but is uncertain about its value when ambiguity is present. Uncertainty about the loss probability is represented by a probability distribution over this loss probability. We define changes in ambiguity over probabilities through a specific case of stochastic dominance of order $n$ as developed by Ekern (1980). This approach makes it possible to define a statistical link between the probability distributions capturing the level of ambiguity over the loss probability. It also makes it possible to link the notion of changes in ambiguity to the properties of the function capturing the individual attitudes towards ambiguity. These properties can be interpreted in terms of preferences for harm disaggregation over probabilities in a similar way as the one developed by Eeckhoudt.
and Schlesinger (2006) in the expected utility theory to explain the meaning of the signs of the successive derivatives of the utility function in terms of preferences for harms disaggregation over wealth. As Eeckhoudt and Schlesinger

So far, increase in ambiguity has been expressed in terms of mean-preserving spread of the loss probability (Snow, 2010, 2011) and in relation to ambiguity aversion in the sense that more ambiguity makes an ambiguity averse individual worse off (Snow, 2010)\(^1\). Our definition of change in ambiguity extends these works and does not necessarily require ambiguity aversion. Indeed, we show that more ambiguity does not necessarily make an ambiguity-averse individual worse off.

Changes in ambiguity over probabilities can be measured in two equivalent ways. The first one, a non-monetary measure, denoted the ambiguity welfare premium, defines the loss of welfare due to facing more ambiguity over probabilities. The second one, a monetary measure, denoted the ambiguity premium, stems from Snow (2010) and corresponds to the willingness to pay to benefit from a reduction in the level of ambiguity over probabilities.

To illustrate our results, we apply the notion of change in ambiguity over probabilities to a problem of optimal resources allocation. We consider a population confronted with a risk of loss for which preventive resource is available. This population differs in the level of ambiguity over the probability of loss. We provide conditions under which a social planner will allocate more resources to population whose probability of loss is more ambiguous.

This paper is organized as follow. In the next section, we introduce the model of ambiguity aversion. In section 3, we present two measures of ambiguity aversion. We then propose the concept of change in ambiguity in section 4. In section 5, an application to optimal resources allocation in the face of more ambiguity is investigated. Finally, a short conclusion is provided in the last section.

## 2 Ambiguity aversion

Let’s consider an individual with an initial wealth \( w \) and confronted with two states of nature, a good state that occurs with probability \((1 - p)\) and a bad state that occurs with probability \( p \) (such that \( 0 < p < 1 \)). The individual expected utility writes as

\[
V_0(w,p) = (1-p)u^G(w) + pu^B(w)
\]

with \( u^{G''}(x) < 0 < u^{G'}(x) \forall x \), \( u^{B''}(x) < 0 < u^{B'}(x) \forall x \) and \( u^G(x) > u^B(x) \forall x \). Utility functions \( u^G \) and \( u^B \) can be either state-independent or state-dependent and could write as follows:

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\(^1\)Changes in ambiguity have recently been studied by Jewitt and Mukerji (2011) in the model of Gilboa and Schmeidler (1982) also in relation to ambiguity aversion.
• $u^G(w) = u(w)$ and $u^B(w) = u(w - L)$ with $L > 0$ and $u'' < 0 < u'$ in the classical model with a monetary loss $L$,

• $u^G(w) = u(w)$ and $u^B(w) = v(w)$ with $u'' < 0 < u'$, $v'' < 0 < v'$, and $u > v$ in the Value of a Statistical Life literature as introduced by Drèze (1962) (where $u$ is the utility of wealth if the individual survives the period, and $v$ is the utility of wealth if the individual dies, that is the utility of leaving a bequest), or in models dealing with an irreplaceable commodity as introduced by Cook and Graham (1977) (where $u$ is the utility in the good state of the world where the irreplaceable commodity is kept, and $v$ is the utility in the other state where the irreplaceable commodity is lost).

Following Treich (2010) and Snow (2011), consider the random variable $\tilde{\epsilon}$, and add it to the probability of the bad state of the world, $p$, so as to represent ambiguity over probabilities\(^2\) with $\tilde{p} = p + \tilde{\epsilon}$. According to KMM (2005), the decision-maker’s utility writes as

$$V(w, p + \tilde{\epsilon}) = \Phi^{-1}\left(E[\Phi\{1 - (p + \tilde{\epsilon})u^G(w) + (p + \tilde{\epsilon})u^B(w)\}]\right) \quad (2)$$

where $E$ denotes the expectation operator over the random variable $\tilde{\epsilon}$. The function $\Phi$ captures the attitude towards ambiguity and is supposed increasing (assuming differentiability), i.e. $\Phi' > 0$\(^3\). The decision maker is considered as strictly ambiguity averse (seeking) if and only if $\Phi$ is strictly concave (convex), as shown by KMM (2005). $\Phi'' < 0$ represents then strict ambiguity aversion, and $\Phi(x) = x$ represents ambiguity neutrality.

In the same way as in the expected utility model, where the addition of $\tilde{\epsilon}$ on the wealth level $w$ reduces the utility of a risk-averse decision maker\(^4\), in the ambiguity model of KMM (2005), the introduction of $\tilde{\epsilon}$ on the probability $p$ reduces the decision maker welfare if he is ambiguity averse, comparing to the case where the probability does not face any ambiguity. Indeed, $\Phi'' < 0$ implies

$$E[\Phi\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\}] < \Phi(E[\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\}]),$$

that is equivalent to (because the function $\Phi^{-1} > 0$ is strictly increasing)

$$\Phi^{-1}\left(E[\Phi\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\}]\right) < V_0(w, p + E(\tilde{\epsilon})) \quad (3)$$

\(^2\)We assume that the realizations of the random variable $\tilde{p} = p + \tilde{\epsilon}$ belong to $[0,1]$. More precisely, we assume that the support of $\tilde{\epsilon}$ is $[\xi, \bar{\epsilon}]$ such that, for all $\epsilon$ in $[\xi, \bar{\epsilon}]$, $p + \epsilon$ verifies $0 < p + \epsilon < 1$.

\(^3\)We implicitly assume that the profitability distribution attached to the random variable epsilon tilde is known. One could question this by stressing that under ambiguity individuals have to form believes over risks and that observing such believe is not straightforward. However, in KMM (2005)’s theory, both first and two stage lotteries are different mathematical concepts and one could expect a different attitude towards the the first and second stage lotteries (see Treich, 2010; Snow, 2011).

\(^4\)Because risk aversion means that $E[u(w + \tilde{\epsilon})] < u(w + E(\tilde{\epsilon}))$. 
that rewrites equivalently (using our notations),
\[ V(w, p + \tilde{\epsilon}) < V_0(w, p + E(\tilde{\epsilon})). \] (4)

Note that for an ambiguity neutral decision-maker, the presence of \( \tilde{\epsilon} \) does not modify the welfare. Indeed, when \( \Phi(x) = x \), we have:
\[ E[\Phi(\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\})] = \Phi(E[\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\})], \]
that is equivalent to
\[ \Phi^{-1}\left(E[\Phi(\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\})]\right) = V_0(w, p + E(\tilde{\epsilon})). \] (5)

### 3 Measuring ambiguity aversion

Following Friedman and Savage (1948), who defined two ways for measuring risk aversion, the utility premium and the risk premium, two ways for measuring ambiguity aversion can be introduced.

By analogy to the utility premium defined by Friedman and Savage (1948) to measure the pain associated with the introduction of a risk on the wealth level\(^5\), we propose to define an “ambiguity welfare premium” as follows:
\[ G(\tilde{\epsilon}) = (1-(p+E(\tilde{\epsilon})))u^G(w)+(p+E(\tilde{\epsilon}))u^B(w) - \Phi^{-1}\left(E[\Phi\{(1 - (p + \tilde{\epsilon}))u^G(w) + (p + \tilde{\epsilon})u^B(w)\})]\right), \]
that is equivalent to (using our notations)
\[ G(\tilde{\epsilon}) = V_0(w, p + E(\tilde{\epsilon})) - V(w, p + \tilde{\epsilon}). \] (6)

The ambiguity welfare premium, \( G(\tilde{\epsilon}) \), measures the pain associated with facing the random variable \( \tilde{\epsilon} \) on the probability \( p \) where pain is defined as a loss of welfare. From eq. (4) it is easy to verify that \( G(\tilde{\epsilon}) > 0 \) if \( \Phi'' < 0 \) since for an ambiguity averse individual the utility is reduced in the presence of ambiguity over the probability \( p \). Note that for an ambiguity-neutral individual, \( G(\tilde{\epsilon}) = 0 \) since by definition a neutral ambiguity decision maker is indifferent between no ambiguity and ambiguity on the probability.

The second way to measure ambiguity aversion is to define a monetary measure corresponding to the amount of money to resolve or eliminate ambiguity as introduced by Snow (2010). Let us denote \( D(\tilde{\epsilon}) \), the willingness to pay to resolve ambiguity over the loss probability, e.g. the willingness to pay to replace \( p + \tilde{\epsilon} \) by its expectation \( p + E(\tilde{\epsilon}) \). \( D(\tilde{\epsilon}) \) is then solution of the following equation:
\[ V(w, p + \tilde{\epsilon}) = (1 - (p + E(\tilde{\epsilon})))u^G(w - D(\tilde{\epsilon})) + (p + E(\tilde{\epsilon}))u^B(w - D(\tilde{\epsilon})), \] (7)

\(^5\)See also Eeckhoudt and Schlesinger (2009) for a recent analysis of the properties of the utility premium.
that rewrites using our notations

\[ V(w, p + \tilde{\epsilon}) = V_0(w - D(\tilde{\epsilon}), p + E(\tilde{\epsilon})). \]  

(8)

\( D(\tilde{\epsilon}) \) is the monetary measure of the agent ambiguity aversion. It is strictly positive for a strict ambiguity averse decision-maker. Indeed, \( \Phi'' < 0 \) implies \( V_0(w, p + E(\tilde{\epsilon})) > V(w, p + \tilde{\epsilon}) \), that rewrites equivalently (using eq. (8)) as \( V_0(w, p + E(\tilde{\epsilon})) > V_0(w - D(\tilde{\epsilon}), p + E(\tilde{\epsilon})) \), which is equivalent to \( D(\tilde{\epsilon}) > 0 \) because \( V_0 \) is strictly increasing with respect to \( w \). In an analogous manner, we can show that ambiguity neutrality means \( D(\tilde{\epsilon}) = 0 \).

These results can be summarised in the following proposition.

**Proposition 1**

Given a random variable \( \tilde{\epsilon} \) capturing ambiguity and given a strictly ambiguity averse (loving) decision-maker (\( \Phi''(x) < (>) 0 \ \forall x \)), the two following items are equivalent:

(i) the ambiguity welfare premium is strictly positive (negative), \( G(\tilde{\epsilon}) > (<) 0 \),

(ii) the ambiguity risk premium is strictly positive (negative), \( D(\tilde{\epsilon}) > (<) 0 \).

4 Changes in ambiguity

While ambiguity aversion expresses preferences for non-ambiguous situation over ambiguous situation, it does not allow to express preferences over two ambiguous situations. A more general issue is to define preferences between two situations of ambiguity over probabilities, i.e. to compare two sources of ambiguity. To that aim, let us consider a strict ambiguity averse decision maker. Let’s define two situations of ambiguity. In the first situation, ambiguity is captured by \( \tilde{\epsilon}_1 \) and in the second one, ambiguity is captured by \( \tilde{\epsilon}_2 \) with \( E(\tilde{\epsilon}_1) = E(\tilde{\epsilon}_2) \). The decision-maker endowed utility writes now as

\[ V(w, p + \tilde{\epsilon}_1) = \Phi^{-1}(E[\Phi\{(1 - (p + \tilde{\epsilon}_1))u^G(w) + (p + \tilde{\epsilon}_1)u^B(w)\}]), \]  

(9)

in the first situation, and it writes as

\[ V(w, p + \tilde{\epsilon}_2) = \Phi^{-1}(E[\Phi\{(1 - (p + \tilde{\epsilon}_2))u^G(w) + (p + \tilde{\epsilon}_2)u^B(w)\}]), \]  

(10)

in the second one. Following eq. (6), ambiguity welfare premia are respectively

\[ G(\tilde{\epsilon}_1) = V_0(w, p + E(\tilde{\epsilon}_1)) - V(w, p + \tilde{\epsilon}_1), \]  

(11)

and

\[ G(\tilde{\epsilon}_2) = V_0(w, p + E(\tilde{\epsilon}_2)) - V(w, p + \tilde{\epsilon}_2). \]  

(12)

From the definition of the ambiguity welfare premium, having \( G(\tilde{\epsilon}_2) > G(\tilde{\epsilon}_1) \) means that the loss of welfare from facing the level of ambiguity captured by \( \tilde{\epsilon}_2 \) is higher than the one
from facing the level of ambiguity captured by \( \hat{e}_1 \), or equivalently that \( \hat{e}_2 \) is considered by the decision maker as more ambiguous than \( \hat{e}_1 \). Our goal is then to define conditions under which \( G(\hat{e}_2) > G(\hat{e}_1) \) or equivalently such as

\[
E[\Phi\{(1-(p+\hat{e}_1))u^G(w)+(p+\hat{e}_1)u^B(w)\}] > E[\Phi\{(1-(p+\hat{e}_2))u^G(w)+(p+\hat{e}_2)u^B(w)\}] \tag{13}
\]

since \( \Phi^{-1'} > 0 \) (because \( \Phi' > 0 \)).

Note that it is equivalent to work either with \( G(\hat{e}) \) or with the monetary measure of ambiguity \( D(\hat{e}) \). Indeed, assuming \( D(\hat{e}_2) > D(\hat{e}_1) \) means that the level of ambiguity captured by \( \hat{e}_2 \) is higher than the one captured by \( \hat{e}_1 \) since the decision-maker is willing to pay more to resolve it. Using eq. (8), investigating the conditions under which \( D(\hat{e}_2) > D(\hat{e}_1) \) is equivalent to investigate the ones under which \( V_0(w-D(\hat{e}_2), p+E(\hat{e}_2)) < V_0(w-D(\hat{e}_1), p+E(\hat{e}_1)) \) (since \( E(\hat{e}_1) = E(\hat{e}_2) \)), that is equivalent to eq. (13).

To model changes in ambiguity between \( \hat{e}_1 \) and \( \hat{e}_2 \), we use a specific case of stochastic dominance of order \( n \) as developed by Ekern (1980). Consider two random variables \( X \) and \( Y \) with \( F \) and \( G \) respectively their two cumulative distribution functions defined over a probability support contained within the open interval \([a, b]\). Define \( F_1 = F \) and \( G_1 = G \).

Now define \( F_{k+1}(z) = \int_a^z F_k(t)dt \) and \( G_{k+1}(z) = \int_a^z G_k(t)dt \) for \( k \geq 1 \). The variable \( Y \) dominates the variable \( X \) to the order of \( n \) in Ekern’s sense \( (X \preceq_{\text{Ekern-}n} Y) \) if \( F_n(z) \geq G_n(z) \) for all \( z \), and if \( F_k(b) \geq G_k(b) \) for \( k = 1, 2, ..., n-1 \) and \( E(X^k) = E(Y^k) \) \( \forall k = 1, 2, ..., n-1 \). The \( n \)-th order dominance in sense of Ekern corresponds to the \( n \)-th order stochastic dominance where the \( (n-1) \) moments of \( X \) and \( Y \) are equal. As an example, Ekern-\( n \) order for \( n = 2 \) and \( n = 3 \) are well-known in expected utility theory. \( X \preceq_{\text{Ekern-}2} Y \) is what Rothschild and Stiglitz (1970) defined as a “mean-preserving increase in risk”. Similarly, \( X \preceq_{\text{Ekern-}3} Y \) is what Menezes et al. (1980) called an “increase in downside risk”.

Recall (from Ingersoll (1987) and according to Ekern (1980)’s definition) the following property: \( X \preceq_{\text{Ekern-}n} Y \) is equivalent to \( E[f(X)] \leq E[f(Y)] \) for any function \( f \) such that \((-1)^{(n+1)} f^{(n)} > 0 \) for all \( n \) such that \( n \geq 1 \).

In proposition 2 below, using Ekern’s definition, we show that an increase in ambiguity can be defined through both the statistical link between \( \hat{e}_2 \) and \( \hat{e}_1 \) and the signs of the higher derivatives of the transformation function \( \Phi \), e.g. \( \Phi', \Phi'', \Phi^{(3)}, \Phi^{(4)}, ..., \Phi^{(n)} \). We pose as an assumption that these derivatives alternate of signs. Indeed, we know that for a strict ambiguity averse decision maker, \( \Phi' > 0 \), and \( \Phi'' < 0 \). We therefore assume that this alternation of signs remains true for higher orders, i.e. \( \Phi^{(3)} > 0, \Phi^{(4)} < 0, \Phi^{(5)} > 0 \), etc, as it is commonly done in the literature dealing with the KMM (2005) model. Indeed, in their original paper KMM (2005) suggest to use the function \( \Phi(x) = \frac{1-\exp(-\alpha x)}{\alpha} \) with
\(\alpha > 0\) as an illustration of their model which exhibits constant ambiguity aversion. It is easy to show that this function is such that \((-1)^{(n+1)}\Phi^n > 0\) for all \(n \geq 1\), i.e. sharing the same properties as the ones exhibited above.

Our main results are given in the following proposition (see the proof in appendix 1):

**Proposition 2**

*Given a strictly ambiguity averse decision-maker with ambiguity attitude captured by \(\Phi\) such that \((-1)^{(n+1)}\Phi^n > 0\) \(\forall n \geq 1\), \(\tilde{\epsilon}_2\) is more ambiguous than \(\tilde{\epsilon}_1\) if and only if one of the three following conditions is verified:

1. \(G(\tilde{\epsilon}_2) > G(\tilde{\epsilon}_1)\)
2. \(D(\tilde{\epsilon}_2) > D(\tilde{\epsilon}_1)\)
3. \(\tilde{\epsilon}_2 \preceq_{\text{Ekern} - n} \tilde{\epsilon}_1\) when \(n\) is even, and \(\tilde{\epsilon}_1 \preceq_{\text{Ekern} - n} \tilde{\epsilon}_2\) when \(n\) is odd.*

Proposition 2 shows that a greater level of ambiguity between \(\tilde{\epsilon}_2\) and \(\tilde{\epsilon}_1\) is not always equivalent to the variable \(\tilde{\epsilon}_2\) being dominated by \(\tilde{\epsilon}_1\) in Ekern’s sense, but can also correspond to the other way round as explained below.

The intuitive explanation of Proposition 2 is the following. If \(\tilde{\epsilon}_1\) and \(\tilde{\epsilon}_2\) are degenerated random variables, \(\tilde{\epsilon}_1 = 0\) and \(\tilde{\epsilon}_2 = k\) with \(k > 0\), then according to the definition of Ekern, \(\tilde{\epsilon}_2\) dominates \(\tilde{\epsilon}_1\) to the order of 1. However, the passage from \(\tilde{\epsilon}_1\) to \(\tilde{\epsilon}_2\) corresponds in our model to an increase in the probability of loss. Thus this passage is considered as an adverse outcome for the decision-maker and any individual such as \(\Phi' > 0\) dislikes this. This explains why the passage from \(\tilde{\epsilon}_1\) to \(\tilde{\epsilon}_2\) constitutes an increase in ambiguity, where more ambiguity corresponds to a higher probability of loss.

If \(\tilde{\epsilon}_1 = 0\) and \(\tilde{\epsilon}_2 = \tilde{\epsilon}\) with \(E(\tilde{\epsilon}) = 0\), then \(\tilde{\epsilon}_1\) dominates \(\tilde{\epsilon}_2\) to the order of 2 in Ekern’s sense. The passage from \(\tilde{\epsilon}_1\) to \(\tilde{\epsilon}_2\) is a mean-preserving spread in the space of probabilities which is disliked by all individuals averse to ambiguity, i.e. such that \(\Phi'' < 0\). Consequently, this passage is an adverse outcome for the decision-maker, which explains why \(\tilde{\epsilon}_2\) is considered as more ambiguous than \(\tilde{\epsilon}_1\), where more ambiguity corresponds to an uncertain probability of loss. Note that this case is equivalent to the definition of increase ambiguity as proposed by Snow (2010).

Let us now consider the case \(n = 3\) with the two following random variables: \(\tilde{\epsilon}_1 = [k, \tilde{\epsilon}; 1/2, 1/2]\) and \(\tilde{\epsilon}_2 = [k + \tilde{\epsilon}, 0; 1/2, 1/2]\). Using the property of Ingersoll (1987), it is easy to verify that \(\tilde{\epsilon}_2\) dominates \(\tilde{\epsilon}_1\) in Ekern’s sense to the order of 3. However, Proposition 2 means that a prudent ambiguity decision-maker (e.g. such as \(\Phi^{(3)}(x) > 0\)) prefers \(\tilde{\epsilon}_1\) to \(\tilde{\epsilon}_2\). Why does \(\tilde{\epsilon}_2\) correspond to more ambiguous than \(\tilde{\epsilon}_1\)? The intuitive explanation is the following. Recall that \(k\) and \(\tilde{\epsilon}\) represent adverse outcomes for the decision-maker. Consequently, a prudent ambiguity decision-maker prefers not to be confronted with these two adverse outcomes together in one state of nature as it is the case with the lottery \(\tilde{\epsilon}_2\). He rather prefers to disaggregate these two adverse outcomes across states of nature as it is the case.
with the lottery \( \tilde{\epsilon}_1 \). Hence, a more ambiguous random variable defined through a specific case of dominance stochastic of order three does not necessarily make an ambiguity averse worse-off since ambiguity aversion is not required.

Such preference for disaggregation of adverse outcomes also applies for higher orders. Indeed, let us consider now the case \( n = 4 \) with the following random variables: \( \tilde{\epsilon}_1 = [\tilde{\theta}, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}] \) and \( \tilde{\epsilon}_2 = [\tilde{\theta} + \tilde{\epsilon}, 0; \frac{1}{2}, \frac{1}{2}] \), with \( E(\tilde{\theta}) = 0 \) and \( \tilde{\theta} \) and \( \tilde{\epsilon} \) are i.i.d. According to the definition of Ekern, \( \tilde{\epsilon}_2 \) is dominated by \( \tilde{\epsilon}_1 \) to the order of 4. Proposition 2 tells us that \( \tilde{\epsilon}_1 \). Indeed, both risks \( \tilde{\epsilon} \) and \( \tilde{\theta} \) are adverse outcomes for the individual. A temperant ambiguity decision-maker (i.e. such as \( \Phi^{(4)}(x) < 0 \)) prefers to disaggregate these adverse outcomes rather than aggregate them, and thus prefers \( \tilde{\epsilon}_1 \) to \( \tilde{\epsilon}_2 \). The random variable \( \tilde{\epsilon}_1 \) is thus less ambiguous than \( \tilde{\epsilon}_2 \).

To summarize, this interpretation is similar to the one developed by Eeckhoudt and Schlesinger (2006) in the expected utility theory to explain the meaning of the signs of the successive derivatives of the utility function in terms of preferences for harms disaggregation. Eeckhoudt and Schlesinger (2006) introduced risk apportionment of order \( n \), \( n = 1, 2, \ldots \), by imposing preferences over simple lotteries\(^6\). These higher-order risk attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable risks and losses. Eeckhoudt and Schlesinger (2006) proved that a differentiable utility function \( u \) satisfies risk apportionment of order \( n \) if, and only if, its successive derivative have alternated signs\(^7\).

In this paper, the signs of the successive derivatives of \( \Phi \) represent preferences for harms disaggregation where harms are defined on probability. We introduce the “ambiguity apportionment” of order \( n \), \( n = 1, 2, \ldots \), that coincides to preferences over simple random variables capturing the ambiguity. These higher-order ambiguity attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms. Proposition 2 proves that a differentiable utility function \( \Phi \) satisfies risk apportionment of order \( n \) if, and only if, it fulfills the condition \((-1)^{(n+1)}\Phi^{(n)}(x) > 0\).

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\(^6\)These lotteries were characterized by Roger (2011) who established that they only differ by their moments of order greater than or equal to \( s \). See also Ebert (2013).

\(^7\)The interested reader can see Eeckhoudt et al.(2009), Eeckhoudt et al. (2010) and Denuit and Rey (2013) for generalizations of this framework.
5 An application to optimal resources allocation

We now illustrate our results via a problem of optimal resources allocation. Most countries need to prioritise resources among their population, especially resources allocated to reduce or compensate the consequences of loss, whether environmental loss (Feldman et al., 1999), health loss (Hoel, 2003) or road-traffic loss (Hokstad and Steiro, 2006) for instance. Such preventive resources need to be allocated before the occurrence of the loss. However knowing the exact probability of loss is difficult and thus the level of ambiguity could be different from one population to another. This application addresses the issue of optimal resources allocation when populations differ in the level of ambiguity over the probability of loss.

Consider a population confronted with two states of nature, a bad state of nature with a loss, either monetary or non-monetary, and a good state of nature with no loss. Preventive activity is available increasing wealth in the state of loss should it occurs. The population consists of two types of individuals identical in all respects except for the fact that the probability of loss is more ambiguous (in the sense of Proposition 2) for the second type of individual than for the first. A proportion $\alpha_1$ has a level of ambiguity over the probability of loss $p + \tilde{\epsilon}_1$, while the remainder of the population, $\alpha_2$, has a level of ambiguity over the probability of loss $p + \tilde{\epsilon}_2$, with naturally $\alpha_1 + \alpha_2 = 1$. We assume that individuals are endowed with initial wealth $w$. Let $c$ denote the level of preventive activity that increases wealth in the bad state of nature where the loss occurs. The productivity of preventive activity is defined through the function $m(c)$ with $m'(c) > 0$ and $m''(c) < 0$ as commonly assumed in the literature\(^8\) (see e.g. Ehlrich and Becker, 1972).

Consider the socially optimal allocation of a fixed budget $r$ that has to be made at the societal level. The risk-averse social planner has a social utility function $u^G$ and $u^B$, respectively in the good state of nature and bad state of nature as defined in section 2. His attitude towards ambiguity is depicted through the function $\Phi(\cdot)$. The social planner must choose the level of resources, $c_1$ and $c_2$, respectively allocated to type-1 and type-2 individuals, with the goal to maximise his expected welfare under the budgetary constraint $r = \alpha_1 c_1 + \alpha_2 c_2$. Should the social planner allocate more or less resources to the individual whose probability of loss is more ambiguous?

The optimisation problem is then represented by the Lagrangian expression $\mathcal{L}$:\(^9\):

$$
\mathcal{L}(c_1, c_2, \lambda) = \alpha_1(E[\Phi\{(1 - \tilde{p}_1)u^G(w) + \tilde{p}_1 u^B(w + m(c_1))\}])
+ \alpha_2(E[\Phi\{(1 - \tilde{p}_2)u^G(w) + \tilde{p}_2 u^B(w + m(c_2))\}]) + \lambda(r - \alpha_1 c_1 - \alpha_2 c_2) \quad (14)
$$

\(^8\) In the classical model of monetary loss, for the bad state of nature, we have $u^B(w + m(c)) = u(w - L + m(c)) = u(w - L(c))$, where $L(c) = L - m(c)$ with $L'(c) = -m'(c) < 0$ and $L''(c) = -m''(c) > 0$ as assumed in Ehrlich and Becker (1972).

\(^9\) Without loss of generality, we omitted the transformation function $\Phi^{-1}$ as usually done in the literature since it does not impact the properties of the solutions.
where \( \tilde{p}_i = p + \tilde{\epsilon}_i, i = 1, 2 \). With an interior solution, the first order conditions imply that the optimal allocations, denoted \( c_1^* \) and \( c_2^* \), satisfy:

\[
E[\tilde{p}_1 m'(c_1^*)u^B(w + m(c_1^*))\Phi\{(1 - \tilde{p}_1)u^G(w) + \tilde{p}_1 u^B(w + m(c_1^*))\}]
= E[\tilde{p}_2 m'(c_2^*)u^B(w + m(c_2^*))\Phi\{(1 - \tilde{p}_2)u^G(w) + \tilde{p}_2 u^B(w + m(c_2^*))\}].
\]  
(15)

Using Ekern’s definition, we obtain the following proposition (see the proof in appendix 2).

**Proposition 3**

Given two types of individuals confronted with the same probability and amount of loss, for which the level of ambiguity on the probability of loss is respectively \( \tilde{\epsilon}_1 \) and \( \tilde{\epsilon}_2 \). The social planner, with ambiguity attitude captured by \( \Phi \) such that \( (-1)^{(n+1)}\Phi^{(n)} > 0 \forall n \geq 1 \), should allocate more resources to type-2 individuals than to type-1 individuals (\( c_2^* \geq c_1^* \)) if \( \tilde{\epsilon}_2 \) is more ambiguous than \( \tilde{\epsilon}_1 \) (in the sense of Proposition 2).

Hence from Proposition 3, we have that a social planner such as \( (-1)^{(n+1)}\Phi^{(n)} > 0 \) \( \forall n \geq 1 \) will allocate more resources to type-2 individuals if \( \tilde{\epsilon}_2 \preceq_{\text{Ekern}\text{-n}} \tilde{\epsilon}_1 \) when \( n \) is even, and if \( \tilde{\epsilon}_1 \preceq_{\text{Ekern}\text{-n}} \tilde{\epsilon}_2 \) when \( n \) is odd.

These results can be illustrated in a way similar as the ones of Proposition 2. Indeed, having \( p + \tilde{\epsilon}_2 \) more ambiguous than \( p + \tilde{\epsilon}_1 \) is equivalent to have \( \tilde{\epsilon}_1 \) dominating \( \tilde{\epsilon}_2 \) to the order of \( n \) when \( n \) is even and \( \tilde{\epsilon}_2 \) dominating \( \tilde{\epsilon}_1 \) to the order of \( n \) when \( n \) is odd. In the case of \( n = 1 \), \( p + \tilde{\epsilon}_2 \) dominates \( p + \tilde{\epsilon}_1 \) to the order of 1 and the passage from \( \tilde{\epsilon}_1 \) to \( \tilde{\epsilon}_2 \) is equivalent to an increase in the probability of loss which is considered also as being more ambiguous than \( p + \tilde{\epsilon}_1 \). In that case, the social planner allocate more resources to the population whose probability of loss is higher. In the case where \( n = 2 \), the social planner allocates more resources to the population whose probability of loss is more ambiguous, where more ambiguity is defined in terms of mean-preserving spread in the space of probabilities.

6 Conclusion

While ambiguity aversion expresses preferences for non-ambiguous situation over ambiguous situation, this paper goes one step further and extends the concept of ambiguity aversion by proposing preferences over more ambiguous probability. Changes in ambiguity over probabilities distributions are expressed through a specific case of stochastic
dominance of order $n$ as developed by Ekern (1980), and can be interpreted in terms of harms disaggregation over probabilities. In particular, a random variable being more ambiguous than another is equivalent to having the first variable being dominated at the order $n$ by the second one in Ekern’s sense when $n$ is even and the second variable being dominated at the order $n$ by the first one in Ekern’s sense when $n$ is odd. This implies that contrary to previous literature, changes in ambiguity are not necessarily defined in relation to ambiguity aversion. Hence, it can happen that more ambiguity does not necessarily make an ambiguity-averse individual worse off.

These results can be useful in many applications where the level of ambiguity is different from one situation to another. We provide an application in terms of optimal resources allocation. We show, under common assumptions on ambiguity attitudes, that a social planner would allocate more resources to reduce a potential loss to individuals confronted with a higher level of ambiguity over the probability of this loss.

There are several limitations of this analysis that need to be pointed out. First, this paper considers only an additive source of ambiguity. We could also express ambiguity impacting the probability in a multiplicative form. Second, we have also considered the model of KMM (2005) to express ambiguity preferences. An extension of this paper would be to consider other models of decisions incorporating attitude towards ambiguity. Yet, it should be stressed that the KMM (2005) model is especially adapted to our analysis since individuals’ subjective beliefs about objective probabilities are represented by a probability distribution which makes it easy to use the specific concept of stochastic dominance of order $n$ as introduced by Ekern (1980). Finally, we have considered a binary risk where ambiguity impacts only one state of nature. A generalization would be to consider ambiguity impacting a continuum state of nature. Extending our work towards these directions would be some interesting topics for future research.
Appendix 1
Proof of proposition 2.

\[ E[\Phi \{ 1 - (p + \tilde{c}_1)u^G(w) + (p + \tilde{c}_1)u^B(w) \}] > E[\Phi \{ 1 - (p + \tilde{c}_2)u^G(w) + (p + \tilde{c}_1)u^B(w) \}] \]
rewrites as:
\[ E[\Phi \{ V_0(w) + \tilde{c}_1(u^B(w) - u^G(w)) \}] > E[\Phi \{ V_0(w) + \tilde{c}_1(u^B(w) - u^G(w)) \}]. \]

Let’s introduce the function \( f \) as follows: \( f(\epsilon) = \Phi \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right) \).

The previous inequality rewrites as: \( E[f(\tilde{c}_2)] < E[f(\tilde{c}_1)] \). We obtain:
\[
\begin{align*}
    f'(\epsilon) &= (u^B(w) - u^G(w))\Phi' \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right), \\
    f''(\epsilon) &= (u^B(w) - u^G(w))^2\Phi'' \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right), \\
    f^{(3)}(\epsilon) &= (u^B(w) - u^G(w))^3\Phi^{(3)} \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right), \\
    f^{(4)}(\epsilon) &= (u^B(w) - u^G(w))^4\Phi^{(4)} \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right), \\
    & \vdots \\
    f^{(n)}(\epsilon) &= (u^B(w) - u^G(w))^n\Phi^{(n)} \left( V_0(w) + \epsilon (u^B(w) - u^G(w)) \right).
\end{align*}
\]

As by assumption, \((u^B(w) - u^G(w)) < 0\), we obtain:
\[
\begin{align*}
    f'(\epsilon) &< 0 \text{ iff } \Phi'(x) > 0 \forall x, \\
    f''(\epsilon) &< 0 \text{ iff } \Phi''(x) < 0 \forall x, \\
    f^{(3)}(\epsilon) &< 0 \text{ iff } \Phi^{(3)}(x) > 0 \forall x, \\
    f^{(4)}(\epsilon) &< 0 \text{ iff } \Phi^{(4)}(x) < 0 \forall x, \\
    & \vdots \\
    f^{(n)}(\epsilon) &< 0 \text{ iff } \Phi^{(n)}(x) > 0 \forall x \text{ when } n \text{ is odd, and} \\
    f^{(n)}(\epsilon) &< 0 \text{ iff } \Phi^{(n)}(x) < 0 \forall x \text{ when } n \text{ is even.}
\end{align*}
\]

Assuming the alternation of signs of the transformation function, and using the definition of \( \text{Ekern} \), we obtain:
if \( \tilde{c}_2 \leq_{\text{Ekern}} \tilde{c}_1 \) for \( n \) even, then \( f^{(n)}(\epsilon) < 0 \) is equivalent to \( E[f(\tilde{c}_2)] < E[f(\tilde{c}_1)] \),
if \( \tilde{c}_1 \leq_{\text{Ekern}} \tilde{c}_2 \) for \( n \) odd, then \( f^{(n)}(\epsilon) < 0 \) is equivalent to \( E[f(\tilde{c}_2)] < E[f(\tilde{c}_1)] \).

Appendix 2
Proof of proposition 3.

Let’s introduce a function \( g \) defined as follows:
\[
g(\epsilon) = m'(c)u^B(w+m(c))(p+\epsilon)\Phi' \{(1-p)u^G(w)+pu^B(w+m(c))+\epsilon(u^B(w+m(c))-u^G(w))\}.
\]

In the goal to simplify the presentation, we adopt the following notations: \( \forall c, D(c) = m'(c)u^B(w+m(c)) \), \( A(c) = (1-p)u^G(w)+pu^B(w+m(c)) \), \( B(c) = u^B(w+m(c))-u^G(w) \).

With these notations, \( g(\epsilon) \) writes as
\[
g(\epsilon) = D(c)(p + \epsilon)\Phi' \{ A(c) + \epsilon B(c) \},
\]
with \( D(c) > 0, A(c) > 0 \), and \( B(c) < 0 \) (because \( m(c) \) is such that \( u^B(w + m(c)) < u^G(w) \) \( \forall c \)). We obtain:

\[
g'(\epsilon) = D(c)\Phi'[A(c) + \epsilon B(c)] + B(c)D(c)(p + \epsilon)\Phi''[A(c) + \epsilon B(c)],
\]
\[
g''(\epsilon) = 2D(c)B(c)\Phi''[A(c) + \epsilon B(c)] + B(c)^2D(c)(p + \epsilon)\Phi'''[A(c) + \epsilon B(c)],
\]
\[
g'''(\epsilon) = 3D(c)B(c)^2\Phi'''[A(c) + \epsilon B(c)] + B(c)^3D(c)(p + \epsilon)\Phi''''[A(c) + \epsilon B(c)],
\]
\[
g^{(4)}(\epsilon) = 4D(c)B(c)^3\Phi^{(4)}[A(c) + \epsilon B(c)] + B(c)^4D(c)(p + \epsilon)\Phi^{(5)}[A(c) + \epsilon B(c)].
\]

The sign alternance of successive derivatives of \( \Phi \) allows to determine the sign of successive derivatives of \( g \). Indeed, we obtain:

if \( \Phi' > 0 \) and \( \Phi'' < 0 \) then \( g'(\epsilon) > 0 \),
if \( \Phi'' < 0 \) and \( \Phi''' > 0 \) then \( g''(\epsilon) > 0 \),
if \( \Phi''' > 0 \) and \( \Phi^{(4)} < 0 \) then \( g'''(\epsilon) > 0 \),
if \( \Phi^{(4)} < 0 \) and \( \Phi^{(5)} > 0 \) then \( g^{(4)}(\epsilon) > 0 \),

... if \( sgn(\Phi^n) = sgn(-\Phi^{n+1}) \) then \( g^n(\epsilon) > 0 \forall n \).

Applying Ekern’s definition, we thus have:

if \( \tilde{\epsilon}_2 \preceq_{Ekern-1} \tilde{\epsilon}_1 \) then \( E[g(\tilde{\epsilon}_2)] \leq E[g(\tilde{\epsilon}_1)] \) with \( g \) such that \( g'(\epsilon) \geq 0 \),
if \( \tilde{\epsilon}_2 \preceq_{Ekern-2} \tilde{\epsilon}_1 \) then \( E[g(\tilde{\epsilon}_2)] \geq E[g(\tilde{\epsilon}_1)] \) with \( g \) such that \( g''(\epsilon) \geq 0 \),
if \( \tilde{\epsilon}_2 \preceq_{Ekern-3} \tilde{\epsilon}_1 \) then \( E[g(\tilde{\epsilon}_2)] \leq E[g(\tilde{\epsilon}_1)] \) with \( g \) such that \( g^{(3)}(\epsilon) \geq 0 \),
if \( \tilde{\epsilon}_2 \preceq_{Ekern-4} \tilde{\epsilon}_1 \) then \( E[g(\tilde{\epsilon}_2)] \geq E[g(\tilde{\epsilon}_1)] \) with \( g \) such that \( g^{(4)}(\epsilon) \geq 0 \), ...

Recall that Proposition 2 tells us that: “\( \tilde{\epsilon}_2 \) is more ambiguous than \( \tilde{\epsilon}_1 \)” \( \iff \tilde{\epsilon}_2 \preceq_{Ekern-n} \tilde{\epsilon}_1 \) if \( n \) is odd and \( \tilde{\epsilon}_2 \preceq_{Ekern-n} \tilde{\epsilon}_1 \) if \( n \) is even.

Let’s consider, for example, the case where \( \tilde{\epsilon}_2 \preceq_{Ekern-2} \tilde{\epsilon}_1 \), that corresponds to the case where \( \tilde{\epsilon}_2 \) is more ambiguous than \( \tilde{\epsilon}_1 \), we have \( E[g(\tilde{\epsilon}_2)] \geq E[g(\tilde{\epsilon}_1)] \) that is equivalent to:

\[
E[D(c_2^*)(p + \tilde{\epsilon}_2)\Phi'[A(c_2^*) + \tilde{\epsilon}_2 B(c_2^*)]] \geq E[D(c_2^*)(p + \tilde{\epsilon}_1)\Phi'[A(c_2^*) + \tilde{\epsilon}_1 B(c_2^*)]]
\]

that is equivalent to (using eq. (15)):

\[
E[D(c_1^*)(p + \tilde{\epsilon}_1)\Phi'[A(c_1^*) + \tilde{\epsilon}_1 B(c_1^*)]] \geq E[D(c_2^*)(p + \tilde{\epsilon}_1)\Phi'[A(c_2^*) + \tilde{\epsilon}_1 B(c_2^*)]].
\]

For a given \( \tilde{\epsilon}_1 \), we define the following function: \( G(c) = E[D(c)(p + \tilde{\epsilon}_1)\Phi'[A(c) + \tilde{\epsilon}_1 B(c)]] \). The previous equation rewrites as \( G(c_1^*) \geq G(c_2^*) \) that is equivalent to \( c_1^* \leq c_2^* \) (because \( G'(c) < 0 \forall c \)).
Let's now consider the case where the random variable $\tilde{\epsilon}_2 \preceq_{Ekern-3} \tilde{\epsilon}_1$, that is $\tilde{\epsilon}_2$ less ambiguous than $\tilde{\epsilon}_1$, we have $E[g(\tilde{\epsilon}_2)] \leq E[g(\tilde{\epsilon}_1)]$ that rewrites $G(c_1^*) \leq G(c_2^*)$ that is equivalent to $c_1^* \geq c_2^*$.

Proof of $G'(c) < 0 \forall c$:

$G'(c) = E[D'(c)(p + \tilde{\epsilon}_1)\Phi'[A(c) + \tilde{\epsilon}_1 B(c)] + D(c)(p + \tilde{\epsilon}_1)(A'(c) + \tilde{\epsilon}_1 B'(c))\Phi''[A(c) + \tilde{\epsilon}_1 B(c)]]$

$D'(c) = m''(c)u^{B'}(w + m(c)) + (m'^2u^{B''}(w + m(c)) < 0 \forall c$

$A'(c) + \epsilon B'(c) = (p + \epsilon)m'(c)u^{B'}(w + m(c)) > 0 \forall \epsilon$, thus $G'(c) < 0$. 

15
References


