Productivity Differences, Public Budgets and Implicit Transfers

Yingchan LUO and Hubert JAYET
EQUIPPE
University Lille 1

HIGHLY PRELIMINARY DRAFT
NOT TO BE QUOTED

Abstract
We set up a model with two asymmetric regions, differing in productivity, population being imperfectly mobile across regions while capital is perfectly mobile. A single utilitarian central authority uses taxes for providing public goods in each region. Looking at the optimal policies, we show that, when differences in productivity levels are exogenous, the first best policy generates an implicit transfer from the richer region to the poorer one as soon as capital is taxed. However, when differences in productivity levels are generated by an agglomeration externality, households are charged a higher tax in the poorer region, with the consequence that, if capital is not taxed, the first best policy generates an implicit transfer from the poorer to the richer region. Capital taxation narrows this implicit transfer.

1. Introduction

Governments use fiscal policies for getting resources they normally use for providing resources to households and firms. In a multi-regional context, the fiscal policy choices result in a spatial repartition of resources generated by tax collection; and choices for the provision of public goods generate a spatial repartition of public expenditures. These two geographical repartitions have no reason to coincide with each other, so that public policies generate transfers of resources across regions. Most of these transfers are implicit, as they do not result from an explicit decision to transfer public resources from one region to another.

The existence of these implicit transfers has been widely acknowledged by scholars. Surprisingly, if research has often been questioning the logic and the consequences of explicit transfers, e.g. systems of grants implemented by the central government, very little attention has been paid to implicit transfers. Their logic, the factors driving them, their consequences are ignored most of the time.
The source of these transfers is often attributed to the fact that central governments cannot differentiate tax payers across regions, charging the same taxes wherever taxpayers are located. However, even if central governments are allowed to differentiate tax levels across regions, their choices may be constrained by the economic consequences of this differentiation, which notably result from the interregional mobility of assets. And interregional implicit transfers may be an unwarranted consequence of these restrictions.

This neglect of implicit transfers may be linked to the fact that most theoretical models of tax policy work in a symmetric environment, all the regions being similar to each other. It is only when we work in an asymmetric environment, some regions being richer and other regions being poorer, that questions of transfers matter. In such an asymmetric setting, two questions appear. First what is the relationship between the spatial allotments of fiscal resources and public expenditures and which factors influence the existence and the amount of implicit public transfers across regions? Second, which is the impact of constraints making these public transfers impossible?

It is these questions we are trying to answer using a rather simple framework where a central planner provides local public goods out of tax resources, in a multiregional economy with two regions. Tax resources come from a personal tax on workers and a linear tax on capital. The two regions differ in their productivity, production using perfectly mobile capital and imperfectly mobile labor. Imperfect mobility of workers is generated by preferences, which generate mobility costs when a worker locates in a region that is not his best choice.

We first look at the central planner’s first best policy when productivity differences between regions are purely exogenous. We show that, at the first best, as soon as capital is taxed, the planner gets tax revenue higher (lower) than local expenditures in the more (less) productive region; then, the public budget generates an implicit transfer from the more productive region to the less productive one, and this transfer is not motivated by equity considerations. If the planner is not allowed to make implicit transfers (local expenditures must be funded out of local tax revenues), the only solution for reaching a first best outcome is not to tax capital.

We first look at the central planner’s first best policy when productivity differences come from agglomeration economics, which make production more efficient in larger regions. The results are strikingly different: the application of the Pigovian principle to the externalities generated by migrants lead to higher taxation of households in the small (and less productive region) than in the large one, and then, if capital is not taxed, the public budget generates a transfer from the less productive region to the most productive one. For mitigating this transfer, one must tax capital.

Despite the different results, the same mechanism is at work. The mobility of capital and households, and externalities, constraint the fiscal choices of the central planner. In the first case, both capital and household tax levels are equalized across regions. In the second case, the application of the Pigovian taxation leads to a fixed interregional
differential in household taxes. These constraints, jointly with the consequences of the asymmetry in productivity on the repartition of capital, generate differences in tax revenues that must be offset by transfers.

The structure of the paper is organized as follows. Section 2 is devoted to the presentation of the model is developed based on a series of economic assumptions. Section 3 examines the case of a purely exogenous asymmetry between regions. Section 4 examines the case of an asymmetry generated by agglomeration economics. Section 5 concludes.

2. Model

In this paper, we consider an economy with two regions, inhabited by a continuum of $L$ inhabitants providing labor and consuming a private good and a publicly provided good. All the inhabitants share the same preferences for both goods. However, they differ from each other with respect to their preferences across regions. Each inhabitant has a specific willingness to pay for residing in region 2 instead of region 1. This willingness to pay is randomly distributed across the whole population.

In each region, the private good is produced from labor and capital using a constant returns to scale technology. Labor is supplied by the individuals living in the region. There is a global fixed stock of perfectly mobile capital $K$, choosing to locate in the region where its return is highest. The property of capital is evenly shared over the whole population.

A planner produces the publicly provided good from the private good, at a constant unit rate of transformation. Production is financed out of taxes. The planner is able to use both a per capita tax on inhabitants and a proportional tax on capital invested in each region. Tax levels may differ across regions.

2.1 Workers

There is a continuum of $L = 1$ workers. $L_1$ workers choose to stay in region 1, $L_2$ workers choose to stay in region 2, with $L_1 + L_2 = L = 1$. Every worker supplies one labor unit and holds a fixed amount of capital, $k$. Therefore, the net income of a worker staying in region $i$ is $w_i + \rho k - \tau_i$, where $w_i$ is the wage in region $i$, $\rho$ is the after tax rate of return to capital and $\tau_i$ is the income tax in region $i$.

This net income is used for consumption of a private monetary good so that, in region $i$, private consumption is

$$c_i = w_i + \rho k - \tau_i$$
Moreover, a worker staying in region \( i \) benefits from consumption of a publicly provided good \( z_i \). Workers are endowed with preferences represented by the following utility function (for worker \( l \)):

\[
V_{l,i} = U(c_{i,l}, z_i) + m_{i,l} \\
U(c_{i,l}, z_i) = c_i + u(z_i)
\]

where

\[
m_{1,l} = \mu_l \\
m_{2,l} = 0
\]

\( \mu_l \) is the willingness to pay of worker \( l \) for staying in region 1 instead of region 2.

Workers differ from each other with respect to their preferences across regions, and then that \( \mu \) varies across workers. The cumulative distribution function of \( \mu \) is

\[
\Lambda(\mu): \mathbb{R} \rightarrow ]0,1[ \\
\text{with } 0 < d\Lambda/d\mu < \infty \text{ and } \Lambda(0) = 0.5.
\]

Therefore, for every \( \mu \in \mathbb{R} \), \( \Lambda(\mu) \) is the number of workers whose willingness to pay for staying in region 1 instead of region 2 is below \( \mu \). The shape of the function \( \Lambda(\mu) \) is represented in Figure 1.

![Figure 1](image)

The inverse function of \( \Lambda(\mu) \) is \( M(l) \): for every \( l \in ]0,1[ \), \( M(l) \) is the maximal value of \( \mu \) for the group of \( l \) workers whose willingness to pay for staying in region 1 instead of region 2 is lowest.

Workers are mobile across regions and choose the location where their utility is highest. Then, the choice of an individual whose willingness to pay \( \mu \) is:

\( i = 1 \) iff \( U(c_{1,l}, z_1) + \mu > U(c_{2,l}, z_2) \Rightarrow \mu > \Delta U \\\n\( i = 2 \) iff \( U(c_{1,l}, z_1) + \mu < U(c_{2,l}, z_2) \Rightarrow \mu < \Delta U \)

where \( \Delta U = U(c_{2,l}, z_2) - U(c_{1,l}, z_1) \). Therefore, \( L_2 = \Lambda(\Delta U) \) workers choose region 2 while \( L_1 = 1 - \Lambda(\Delta U) \) choose region 1. Equivalently, for \( L_2 \) workers to choose region 2, the utility differential must be \( \Delta U = M(L_2) \).
2.2 Production of the private good

The private good is produced by private firms combining labor and capital using a constant return to scale technology. In region $i$, the production function is

$$F_i(K_i, L_i) = \theta_i \frac{F(K_i, L_i)}{L_i} K_i f(k_i)$$

where $K_i$ is the capital input, $L_i$ is the labor input, $k_i = K_i/L_i$, $F(K_i, L_i)$ is an homogenous production function while $f(k_i)$ is a concave increasing function meeting the Inada conditions; $\theta_i$ is an efficiency parameter, which differs across regions. Without loss of generality, we assume that $\theta_1 = 1$ and that production is more efficient in region 2, so that $\theta_2 = \theta > 1$.

Sometimes, we will need the following property:

*Single crossing property:* Whatever $\theta > 1$ and $\delta > 0$, the curves $y = f'(k)$ and $y = \theta f'(k + \delta)$ cross once.

There is a fixed capital stock of capital available for production in both regions, $K$, so that

$$K_1 + K_2 = L_1k_1 + L_2k_2 = K$$

Capital is perfectly mobile across region and then the post-tax return to capital, $\rho$, is the same in both regions:

$$\rho = f'(k_1) - t_1 = \theta f'(k_2) - t_2$$

Where $t_i$ is the tax rate on capital in region $i$.

2.3 Public good provision

We assume the publicly provided good to be divisible\(^1\) and that there are no spillovers: for every consumer in region $i$ to consume the quantity $z_i$ of publicly provided goods, the government must provide the global quantity $Z_i = L_iz_i$ in the region. The publicly provided good is produced from the private good with a one-to-one rate of transformation.

A single planner provides the good in both regions, using both the tax on households and the tax on capital. The total amount of taxes collected in region $i$ is

$$L_i\tau_i + K_it_i = L_i(\tau_i + k_it_i)$$

where $\tau_i$ is the amount of the tax on workers and $t_i$ is the rate of the tax on capital.

The central planner balances his whole budget:

$$L_1z_1 + L_2z_2 = L_1(\tau_1 + k_1t_1) + L_2(\tau_2 + k_2t_2)$$

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\(^1\) This assumption has been made because, with an indivisible public good, the largest region has an advantage in the provision of public good. We do not want our results to be driven by this advantage.
2.4 Welfare

The central planner maximizes the total welfare for all the residents in both regions. Then he maximizes the standard utilitarian welfare function:

$$ W = \int_0^1 \left[ c_{i(l)} + u(z_{i(l)}) + m_{i(l),i} \right] dl $$

where $i(l)$ is the location of agent $l$.

Straightforwardly, as workers are homogenous with respect to their contribution to production and the utility they derive from consumption, all the workers staying in region 1 must have a higher willingness to pay for staying in that region than the workers staying in region 2. Then, if there are $L_1$ workers in region 1 and $L_2$ workers in region 2 (with $L_1 + L_2 = 1$), for all the workers staying in region 1 (resp. region 2) we have $\mu > M(L_2)$ (resp. $\mu < M(L_2)$). Then, knowing that $m_{1,i} = \mu_i$ and $m_{2,i} = 0$, we can rewrite $W$ as:

$$ W = L_1[c_1 + u(z_1)] + L_2[c_2 + u(z_2)] + \int_{L_2}^1 M(l) dl $$

which, up to the constant $\int_{L_2}^1 M(l) dl$, may also be written as:

$$ W = L_1[c_1 + u(z_1)] + L_2[c_2 + u(z_2)] - \Omega(L_2) $$

where $\Omega(L_2) = \int_{0.5}^{L_2} M(l) dl$

Note $\Omega(L_2)$ may be interpreted as an aggregated migration cost. Let us assume that, initially, all the workers are located in the region they prefer: the 0.5 migrants with $\mu > 0$ are in region 1 while the 0.5 workers with $\mu < 0$ are in region 2. Then, taking account of the utility differential generated by consumption, $\Delta U$, workers relocate. For a worker with willingness to pay $\mu$, the migration cost is $|\mu|$. And, aggregating over all the migrating workers, we get the aggregated migration cost $\Omega(L_2)$.

3. Central planner taxation choice

3.1 First best optimum

At a first best optimum, the central planner determines the repartitions of population and capital across regions and the levels of public and private consumption, maximising $W$ under the following constraints:

$$ L_1 f(k_1) + L_2 \theta f(k_2) = L_1(c_1 + z_1) + L_2(c_2 + z_2) $$

(2)

$$ L_1 k_1 + L_2 k_2 = k $$

(3)

$$ L_2 = L - L_1 $$

(4)

The first constraint is the budget constraint: the global level of production equals the global level of consumption. The second and third constraints describe the spatial repartitions of the global fixed stocks or capital and population.
In appendix 1, we prove the following proposition:

**Proposition 1:** at the optimal outcome the following equalities hold:
\[
\begin{align*}
\theta f'(k_2) &= f'(k_1) \\
u'(z_1) &= u'(z_2) = 1 \\
M(L_2) &= \theta [f(k_2) - k_2 f'(k_2)] - [f(k_1) + k_1 f'(k_1)]
\end{align*}
\]

The first equality tells us that, at the first best optimum, the marginal productivity of capital is equalized across regions.

The second equality is the standard Samuelson rule: in both regions, the marginal rate of substitution of the private good to the publicly provided good equals the marginal rate of transformation, which equals unity. Together with the assumption that \(u(z_i)\) meets the Inada conditions, it implies that both regions should provide the same amount of public good, \(z_1 = z_2 = z\), with \(u'(z) = 1\).

As for the third equation, we know that \(M(L_2)\) is the willingness to pay of the marginal migrant. \(\theta [f(k_2) - k_2 f'(k_2)]\) is the marginal productivity of labor in region 2, while \(f(k_1) - k_1 f'(k_1)\) is the marginal productivity of labor in region 1. Therefore, the third condition tells us that the marginal migrant (who can be located in either region) has a willingness to pay for staying in region 1 equal to the productivity differential between region 2 and region 1: the preference for region 1 exactly compensates the loss in productivity.

### 3.2 Equilibrium and implementation

In this section, we look at the possibility to implement the first best optimum defined above as equilibrium in an economy where the central planner uses the per capita tax on labor and the proportional tax on capital as policy instruments. The central planner starts, choosing his tax policy. Then, the workers freely choose their location and capital freely splits across regions, taking account of the taxes.

For region \(i\), the central planner levies taxes on capital, \(t_i\), and on households, \(\tau_i\). Capital and workers choose their own location based on the taxes. At equilibrium, the following conditions are met:

\[
\begin{align*}
\rho &= f'(k_1) - t_1 = \theta f'(k_2) - t_2 \\
Z_1 + Z_2 &= L_1 z_1 + L_2 z_2 = L_1 (\tau_1 + k_1 t_1) + L_2 (\tau_2 + k_2 t_2) \\
M(L_2) &= c_2 + u(z_2) - c_1 - u(z_1)
\end{align*}
\]

The first equation tells us that when capital is perfectly mobile, the after tax returns to capital must be the same across regions.

The second equation is the global public budget constraint faced by the planner.
The third equation tells us that, when migrants are perfectly mobile, the marginal migrant is indifferent living in either region, so that his marginal willingness to pay, $M(L_2)$, equals his utility differential: $c_2 + u(z_2) - c_1 - u(z_1)$.

Furthermore, consumption is determined by the private budget constraints:

$$c_1 = f(k_1) - k_1 f'(k_1) + \rho k - \tau_1$$

$$c_2 = f(k_2) - k_2 f'(k_2) + \rho k - \tau_2$$

Let us now compare the conditions for equilibrium and the conditions for a first best optimum. Propositions 2, 3, 4 are proved in appendix 2:

**Proposition 2:** For equalizing marginal productivities, the planner has to tax capital at the same rate in both regions: $t_1 = t_2 = t$

At the first best optimum, the marginal productivity of capital must be same in both regions. On the other hand, at equilibrium with perfectly mobile capital, the post-tax returns to capital must be the same in both regions. Therefore, for equilibrium to coincide with the first best optimum, the central planner has to tax capital at the same rate in both regions.

**Proposition 3:** The choice made by the marginal migrant implies that he must be charged the same head tax in both regions: $\tau_1 = \tau_2 = \tau$

At equilibrium, the marginal migrant equalizes his willingness to pay to the utility differential between regions. In order for this utility differential not to be modified by taxation, head taxes must be the same in both regions.

It’s easy to see the central planner can implement the first best outcome as an equilibrium for every tax package equalizing both taxes in both regions ($\tau_1 = \tau_2 = \tau$ and $t_1 = t_2 = t$) and meeting the public budget constraint, which in that case becomes

$$z = \tau + kt$$

where $z$ is the optimal quantity of the publicly provided good, which is the same in both regions ($z_1 = z_2 = z$, with $u'(z) = 1$). Then, the planner can choose any level of one of the taxes, say $t_1 = t_2 = t$ (resp. $\tau_1 = \tau_2 = \tau$), and choose the other tax, say $\tau_1 = \tau_2 = \tau$ (resp. $t_1 = t_2 = t$), so as to meet his budget constraint. Comparing cost of providing the public goods in both regions and the revenue of taxation, we get the following proposition (see the proof in appendix 3):

**Proposition 4:** If equilibrium implements the first best optimum and capital is positively taxed, more taxes per capita will be collected in the more productive region than in the less productive region, so that providing the same level of public good in
both regions implies a transfer of resources from the more productive to the less productive one.

3.3 Single planner case with no public transfer

Since proposition 4 implies that there may be an implicit transfer between regions, what happens when this transfer is not allowed? Let us first note that if $t_1 = t_2 = 0$ then $z_1 = z_2 = z = \tau_1 = \tau_2$, so that the head tax collected in each region exactly covers the cost of the publicly provided good and there is no transfer between regions. Then an equilibrium where the publicly provided good is funded out of the head tax only does not need any transfer. Moreover, this equilibrium is the only one implementing the first best outcome without transfers, for a simple reason: we proved in Proposition 4 that, for every strictly positive level of capital taxation, at the first best outcome, there is an implicit transfer. Then, not allowing transfers generates a binding constraint which prevents implementing the first best outcome as soon as capital is taxed.

Let us note that this necessity of not taxing capital when we do not allow for public interregional transfers differs from the standard result that decentralized jurisdictions must not tax perfectly mobile capital. In our model, capital is perfectly mobile across jurisdictions, obliging the central planner to tax capital at the same rate in both regions for capital taxation not to be distortive. But the global capital stock is fixed and then, at an efficient outcome, the common tax rate is undetermined. It is explicitly for not generating transfers that capital must not be taxed, because capital is concentrated in the most productive region and, jointly with the obligation to tax capital at the same rate in both concentration, as soon as capital is effectively taxed, this concentration of the tax based generates a concentration of the tax revenue that must be compensated by an implicit transfer.

3.4 Limits on the tax on capital

What happens if the planner is obliged to tax capital? If the planner is allowed to make implicit transfers between regions, the obligation to tax capital does not prevent the planner to implement a first best outcome. However, as noted above, implementing the first best outcome becomes impossible without transfers if the tax on capital is strictly positive. Looking for a second best optimum, we find the following proposition, proved in Appendix 4:

**Proposition 5:** If the central planner is obliged to set a strictly positive tax on capital, $t_1 = t_2 = t > 0$, he still provides the optimal amount of public good in both regions ($z_1 = z_2 = z$, with $u'(z) = 1$), which leads him to levy a higher labor tax in region 1 than in region 2. There are more inhabitants in the more productive region than at the
first best outcome and the unevenness between regions is larger the higher the capital tax rate.

Then, being obliged to tax capital, the planner chooses a second best policy which exacerbates the inequalities between regions.

3.5 Limits on the head tax
Then we consider the symmetric case: what happens if the planner faces an upward limit on household taxation, implying that he cannot finance the whole cost of the public goods out of household taxes? We still look at the case where the planner is not allowed to make transfers between regions. The constraint on household taxation is:
\[ \tau_1 \leq \bar{\tau} \]
\[ \tau_2 \leq \bar{\tau} \]
with \( \bar{\tau} < z \), where \( z \) is the first best level of public good \((u'(z) = 1)\). In Appendix 5, we prove the following proposition:

**Proposition 6**: If the planner faces an upward limit on household taxation, \( \bar{\tau} \), implying that he cannot finance the whole cost of the public goods out of household taxes, in both regions he underprovides the public good with respect to the first best. Moreover, a tighter limitation \((d\bar{\tau} < 0)\) implies:
- Higher capital tax rates in both regions \((dt_1 > 0, dt_2 > 0)\)
- A higher increase of the capital tax rate in the less productive region \((dt_1 - dt_2 > 0)\).
- A lower endowment in capital per worker in the less productive region \((dk_1 < 0)\)
- A lower level of public good provision in the less productive region \((dz_1 < 0)\)

We are unable to sign the impact on capital per worker and public good provision in the most productive region.

4. Agglomeration externalities

In this chapter, we no longer consider that the total factor productivity in each region is fixed. We look at the case where there is an agglomeration externality, the total factor productivity being an increasing function of the population of the region.

4.1 The agglomeration externality
We introduce an agglomeration externality in a very simple way: in each region, the efficiency parameter, \( \theta_i \), is an increasing function of the size of the labor force:
\[ \theta_i = \theta(L_i) \]
where \( \theta \) is a convex increasing function: \( \theta'(L_i) > 0 \) and \( \theta''(L_i) > 0 \). Note that, now, there is no longer any ex-ante asymmetry between the two regions: if \( L_1 = L_2 = 1/2 \), then \( \theta_1 = \theta_2 = \theta(1/2) \). However, if agglomeration externalities are strong enough, mobile workers may tend to agglomerate in one region, leading to an ex-post asymmetric outcome.
4.2 Welfare analysis with an agglomeration externality

The maximization problem determining the first best outcome is the same as in section 4.1. The central planner chooses the repartitions of population and capital across regions and the levels of public and private consumption, maximizing $W$ under the constraints (1), (2) and (3). The only difference is that now, we have to take account of the fact that, in each region, total factor productivity depends upon population, $\theta_i = \theta(L_i)$.

The characteristics of the optimal outcome are derived in Appendix 6. We again find the equality of marginal productivities of capital:

$$\theta_1 f'(k_1) = \theta_2 f'(k_2)$$

and the Samuelson rule:

$$u'(z_1) = u'(z_2) = 1 \Rightarrow z_1 = z_2$$

However, the condition for an optimal repartition of the population changes. It is now:

$$M(L_2) = \theta_2 [f(k_2) - k_2 f'(k_2)] - \theta_1 [f(k_1) - k_1 f'(k_1)] + [L_2 \theta_2' f(k_2) - L_1 \theta'_1 f(k_1)]$$

Compared to the expression of Proposition 1, we now have an additional term, $L_2 \theta'_2 f(k_2) - L_1 \theta'_1 f(k_1)$. What happens is that, compared to the case of fixed total factor productivities, the introduction of agglomeration externalities generates an additional effect of migration. A migrant moving from region 2 to region 1 generates a decrease in the production of workers located in region 2, measured by $L_2 \theta'_2 f(k_2)$, and in increase in region 1, measured by $L_1 \theta'_1 f(k_1)$, hence a productivity differential $L_1 \theta'_1 f(k_1) - L_2 \theta'_2 f(k_2)$. Therefore, the willingness to pay of the marginal migrant $M(L_2)$, has now to compensate for both the differential in marginal productivity of the migrant and his impact on the production of the economy.

Let us note that the symmetric outcome ($L_1 = L_2 = 0.5$ and $k_1 = k_2 = k$) meets the first order conditions. However, it may not be an optimum, because of the non-concavity induced by the externality. Then, beyond the first order conditions, we also have to look at second order conditions. Maximizing $W$ with respect to $k_1$, $k_2$, $z_1$ and $z_2$ for given $L_2$, one finds the welfare function $W(L_2)$. In appendix, we prove that, for $L_1 = L_2 = 0.5$:

$$\frac{\partial^2 W(L_2)}{\partial L_2^2} = \frac{2 \theta' f(k)}{A} \left[ A^2 + 2(1 - \eta)A + \frac{\eta^2}{4k^2} \right]$$

where $A = L_1 \theta'' = 0.5 \theta''$ is the elasticity with respect to population of the marginal increase in the agglomeration externality and $\eta = \frac{k f'(k)}{f(k)}$ is the elasticity of production function with respect to capital. The ratio $\frac{2 \theta' f(k)}{A}$ being positive, $\frac{\partial^2 W(L_2)}{\partial L_2^2}$ has the same sign has the bracketed term, which is a quadratic expression. The determinant of this quadratic expression is
\[ \Delta^2 = 4(1 - \eta)^2 - \frac{\eta^2}{k^2} = \left[ \left( 2 + \frac{1}{k} \right) \eta - 2 \right] \left[ \left( 2 - \frac{1}{k} \right) \eta - 2 \right] \]

Two cases must be distinguished:

- When \( \Delta^2 < 0 \) \( \iff \eta \in \left[ \frac{2k}{2k+1}, \frac{2k}{2k-1} \right] \), \( \frac{\partial^2 W}{\partial l z^2} > 0 \) whatever \( A \).
- When \( \Delta^2 > 0 \) \( \iff \eta \notin \left[ \frac{2k}{2k+1}, \frac{2k}{2k-1} \right] \), \( \frac{\partial^2 W}{\partial l z^2} \leq 0 \) for \( A \in [A_1, A_2] \) and \( \frac{\partial^2 W}{\partial l z^2} > 0 \) for \( A \notin [A_1, A_2] \), with \( A_1 = 2(\eta - 1) - \Delta \) and \( A_2 = 2(\eta - 1) + \Delta \)

Then, as soon as the elasticity with respect to population of the marginal increase in the agglomeration externality, \( A \), is high enough \((A > A_2)\), we are sure that \( \frac{\partial^2 W}{\partial l z^2} > 0 \) whatever \( \eta \), and then the symmetric outcome is a local minimum. The first best outcome is necessarily asymmetric. From now on, we will focus on asymmetric outcomes.

### 4.3 Equilibrium and implementation

As in section 4.3, we look at the possibility to implement the first best optimum defined above as equilibrium in an economy where the central planner uses the per capita tax on labor and the proportional tax on capital as policy instruments. At equilibrium, the conditions (5), (6) and (7) defined in section 4.3 still hold.

In Appendix 7, we prove the following proposition:

**Proposition 7:** A policy implementing the first best optimum is characterized by:

\[ z_1 = z_2 = z \ 	ext{with} \ u'(z) = 1 \]
\[ t_1 = t_2 = t \]
\[ \tau_1 - \tau_2 = \Delta \tau \equiv L_2 \theta' f(k_2) - L_1 \theta' f(k_1) \]
\[ z = L_1 \tau_1 + L_2 \tau_2 + kt \]

Comparing to the case of fixed total factor productivities (Propositions 2 and 3), there is an important difference: the planner no longer equalizes the taxes paid by households across regions. When total factor productivities were fixed, taxes were equalized so as not to distort the location choice of the marginal migrant. As noted above, when there are agglomeration externalities, migration generates externalities, as it decreases the productivity of all the workers located in the origin region and increases the productivity of all the workers located in the destination region. For a move from region 1 to region 2, the net effect of these externalities is \( L_2 \theta' f(k_2) - L_1 \theta' f(k_1) \). Following the Pigovian rule, the differential in household taxes must compensate for these externalities, hence the equality \( \tau_1 - \tau_2 = L_2 \theta' f(k_2) - L_1 \theta' f(k_1) \).

When the first best outcome to be implemented is asymmetric, this application of the Pigovian principle has an important, and maybe undesirable, feature. Without loss of
generality, let us assume that the largest region is region 2, so that $L_2 > 0.5 > L_1$, which implies $\theta_2 > \theta_1$ and $\theta_2' > \theta_1'$; and, using the equality $\theta_1 f'(k_1) = \theta_2 f'(k_2)$, $k_2 > k > k_1$, leading to $f(k_2) > f(k_1)$. Then,
\[ \tau_1 - \tau_2 = \Delta \tau > 0 \]
The implication of this inequality is that households leaving in the smaller and poorer region (Region 1) pay more taxes than households leaving in the larger and richer region (Region 2). This result is a direct consequence of the application of the Pigovian principle: a migrant moving from the largest and richest region to the smallest and poorest one generates a larger decrease in the production of the large region than the increase generated in the smaller region. Then, his global effect is negative and, for compensating this negative effect, he has to pay higher taxes in the smaller and poorer region. However, from an equity point of view, it may be undesirable, a point we will be looking at more closely in the next section.

Let us also note that the planner has one degree of freedom: he can choose $t$ ($= t_1 = t_2$) arbitrarily and, once $t$ has been chosen, $\tau_1$ and $\tau_2$ are determined by the last two equations of Proposition 7, hence:
\[ \tau_1 = z - kt + (L_2)^2 \theta_2 f(k_2) - L_1 L_2 \theta_1 f(k_1) \]
\[ \tau_2 = z - kt - L_1 L_2 \theta_2 f(k_2) + (L_1)^2 \theta_1 f(k_1) \]

### 4.4 Transfers

The per capita tax revenue in each region is
\[ R_1 = \tau_1 + k_1 t = z - (k - k_1)t + L_2 \Delta \tau \]
\[ R_2 = \tau_2 + k_2 t = z - (k - k_2)t - L_1 \Delta \tau \]
with $\Delta \tau = L_2 \theta_2 f(k_2) - L_1 \theta_1 f(k_1)$

As noted above, when the first best outcome to be implemented is asymmetric, region 2 being the largest one, we have:
\[ \tau_1 - \tau_2 = \Delta \tau > 0 \]
so that households leaving in the smaller and poorer region (Region 1) pay more taxes than households leaving in the larger and richer region (Region 2).

Then, if the planner chooses not to tax capital ($t = 0$), $R_1 = \tau_1 > \tau_2 = R_2$ and per capita tax revenue is higher in the smaller and poorer region 1 than in the larger and richer region 2, generating an implicit transfer from the poorer region to the richer one. However:
\[ \frac{dR_1}{dt} = k_1 - k < 0 \]
\[ \frac{dR_2}{dt} = k_2 - k > 0 \]
so that introducing taxation of capital decreases the per capita tax revenue in the poorer region and increases it in the richer one, reducing the level of implicit transfers.
from the poorer to the richer region: the central planner uses the fact that the capital is disproportionately located in region 2. The transfers will be reversed if \( R_1 \leq R_2 \), which implies
\[
t \geq \frac{\Delta \tau}{k_2 - k_1}
\]

5. Conclusion

The central message of our analysis is a fairly simple one: in an economy where there are several regions, even if he is able to differentiate tax levels or tax rates between regions, efficiency considerations may limit the freedom of the central planner to use this differentiated taxation. These limits may come from interregional mobility of tax bases, as taxation must not distort the location choice of data base and/or may need to internalize externalities generated by the mobility of tax bases.

Then, for the central planner, the mobility of tax bases is constraining in two ways: it determines the interregional repartition of tax bases and it constraints the choice of tax levels or tax rates charged on these tax bases. The result of this double constraint is that the spatial repartition of tax revenues may not coincide with the spatial repartition of spatial expenditures, generating implicit transfers in the welfare system. The nature and the direction of these transfers depend upon the type of constraints imposed on the planner. In some cases, these implicit transfers may have undesirable characteristics, for example when a poor regions pay for a rich one.

However, despite the constraint he faces, the planner may still have some degrees of freedom in the choice of the tax menu. For example, in the model of this paper, the planner is still able to choose the repartition of tax revenues between capital taxation and household taxations. The planner may be using this menu for manipulating implicit transfers. For example, in the examples of this paper, the concentration of capital in the most productive region and the constraint that taxes on capital must be equalized across regions generate concentrated tax revenues in the richest region. Then, lower taxation of capital lowers the impact of the concentration of capital on interregional transfers. In the first situation described in this paper (exogenous productivity), this leads the planner not to tax capital if he is unable to make transfers. On the contrary, in the second situation, the planner may want to use capital taxation for compensating the distortion resulting from the fiscal internalization of agglomeration externalities generated by migration.
Appendix 1

The central planner solves the following problem:

\[
\begin{align*}
\text{Max } & L_1[c_1 + u(z_1)] + L_2[c_2 + u(z_2)] - \Omega(L_2) \\
\text{s.t } & L_1 f(k_1) + L_2 \theta f(k_2) = L_1(c_1 + z_1) + L_2(c_2 + z_2) \\
& L_1 k_1 + L_2 k_2 = k \\
& L_1 + L_2 = 1
\end{align*}
\]

Hence the Lagrangian:

\[
Y_1 = (1 - L_2)[c_1 + u(z_1)] + L_2[c_2 + u(z_2)] - \Omega(L_2) + \lambda[(1 - L_2)(f(k_1) - c_1 - z_1) + L_2(\theta f(k_2) - c_2 - z_2)] - \xi[(1 - L_2)k_1 + L_2k_2 - k]
\]

Differentiating the Lagrangian with respect to \( z_1, z_2, c_1, c_2, k_1, k_2 \) and \( L_2 \), and rearranging, we get the following first order conditions:

\[
\begin{align*}
& u'(z_1) = u'(z_2) = 1 \\
& f'(k_1) = \theta f'(k_2) \\
& M(L_2) = \theta f'(k_2) - f'(k_1) - \xi(k_2 - k_1) \\
& \quad = \theta[f(k_2) - k_2f'(k_2)] - [f(k_1) - k_1f'(k_1)]
\end{align*}
\]

Appendix 2:

Looking at firms, at equilibrium with perfectly mobile capital, post tax returns to capital are equalized across regions:

\[
\rho = f'(k_1) - t_1 = \theta f'(k_2) - t_2
\]

We know that, at the first best optimum, marginal productivity is the same in both regions.

\[
\theta f'(k_2) = f'(k_1)
\]

For both equalities to hold simultaneously, we need to have \( t_1 = t_2 \)

Now, looking at households, at equilibrium, the willingness to pay of the marginal households equals the utility differential:

\[
M(L_2) = c_2 + u(z_2) - c_1 - u(z_1)
\]

Knowing that at the first best optimum, \( z_1 = z_2 \), and using the expressions for consumption, we get:

\[
M(L_2) = c_2 - c_1 = \theta[f(k_2) - k_2f'(k_2)] - [f(k_1) - k_1f'(k_1)] + \tau_1 - \tau_2
\]

Furthermore, from Proposition (1), we know that

\[
M(L_2) = \theta[f(k_2) - k_2f'(k_2)] - [f(k_1) - k_1f'(k_1)]
\]

For an equilibrium to implement the first best optimum, both equalities must hold simultaneously, hence \( \tau_1 = \tau_2 \). The planner charges the same taxes on labor in both regions.

Appendix 3:

Knowing that \( \theta > 1 \), \( f'(k_1) = \theta f'(k_2) \) implies \( f'(k_1) > f'(k_2) \) and then, knowing that \( f''(k_1) < 0 \), \( k_1 < k_2 \). Moreover, \( k_1 < k_2 \) and the equalities \( k = L_1k_1 + L_2k_2 \) and \( L_1 + L_2 = 1 \) imply:

\[
k_1 < k < k_2
\]
Tax revenue per capita is $R_1 = \tau_1 + k_1 t_1$ in region 1 and $R_2 = \tau_2 + k_2 t_2$. The central planner charging the same tax rate at capital $t_1 = t_2 = t$ and the same head tax $\tau_1 = \tau_2 = \tau$, we get:

$$t > 0 \Rightarrow R_1 = \tau + k_1 t < \tau + k_2 t = R_2$$

Moreover, knowing that $z_1 = z_2 = z$, the global public budget constraint implies:

$$z = \tau + kt$$

Which, knowing that $k_1 < k < k_2$, implies:

$$t > 0 \Rightarrow R_1 < z < R_2$$

Then, as soon as the central planner taxes capital, per capita tax revenue in region 1 is not high enough for covering per capital public expenditures, the difference being $R_1 - z < 0$; at the same time, per capita tax revenue in region 2 is higher than capital public expenditures, the difference being $R_2 - z > 0$. Then, the planner implicitly transfers tax revenue from region 2 to region 1.

**Appendix 4**

Now we consider the case where the central planner cannot tax capital at a lower rate than $t$, this constraint being binding in both regions, so that $t_1 = t_2 = t$.

When transfers between regions are not allowed, the planner’s second best program maximizes $W$ under the following constraints:

$$c_1 + z_1 = f(k_1) + (k - k_1)(f'(k_1) - t)$$
$$c_2 + z_2 = \theta f(k_2) + (k - k_2)(\theta f'(k_2) - t)$$
$$f'(k_1) - t = \theta f'(k_2) - t \Rightarrow f'(k_1) = \theta f'(k_2)$$
$$M(L_2) = c_2 + u(z_2) - c_1 - u(z_1)$$
$$L_1 k_1 + L_2 k_2 = k$$

The Lagrangian of this program is:

$$\mathcal{L} = L_1 [c_1 + u(z_1)] + L_2 [c_2 + u(z_2)] - \Omega(L_2) - \lambda_1 [c_1 + z_1 - f(k_1) - (k - k_1)(f'(k_1) - t)] - \lambda_2 [c_2 + z_2 - \theta f(k_2) - (k - k_2)(\theta f'(k_2) - t)] - \mu [L_1 k_1 + L_2 k_2 - k] - \zeta [\lambda_1 c_1 + \lambda_2 c_2 + \zeta [c_2 + u(z_2) - c_1 - u(z_1) - M(L_2)]]$$

Differentiating the Lagrangian with respect to $z_1$, $z_2$, $c_1$, $c_2$, $k_1$, $k_2$ and $L_2$, and rearranging, we get the following first order conditions:

$$\xi = L_2 - \lambda_2 = \lambda_1 - L_1 \Rightarrow \lambda_1 = \xi + L_1 \quad \lambda_2 = L_2 - \xi$$
$$u'(z_1) = u'(z_2) = 1 \Rightarrow z_1 = z_2 = z \text{ with } u'(z) = 1$$
$$\mu L_1 = \lambda_1 [t + (k - k_1)f''(k_1)] - \zeta f''(k_1)$$
$$\mu L_2 = \lambda_2 [t + (k - k_2)\theta f''(k_2)] + \zeta \theta f''(k_2)$$
$$M(L_2) - \xi M'(L_2) = c_2 - c_1 - \mu (k_2 - k_1)$$

Note that, when $z_1 = z_2 = z$, the migration constraint becomes $M(L_2) = c_2 - c_1$ and then the last condition becomes

$$\xi M'(L_2) = \mu(k_2 - k_1)$$

The planner is still following the Samuelson rule, providing the same quantity of public good in both regions, $z_1 = z_2 = z$ with $u'(z) = 1$. Then,

$$z = \tau_1 + k_1 t = \tau_2 + k_2 t \Rightarrow \tau_1 - \tau_2 = (k_2 - k_1) t > 0$$
the later inequality being the consequence of the fact that capital is charge the same
tax rate in both regions, which implies $f'(k_1) = \theta f'(k_2)$ and then, $\theta$ being above
unity, $k_1 < k_2$.

The first order conditions imply that, for any given $t, k_1, k_2$ and $L_2$ solve the
following system of equations:

$M(L_2) = \theta f'(k_2) - f(k_1) - (k_2 - k_1)(f'(k_1) - t)$

$\theta f''(k_2) = f''(k_1)$

$(1 - L_2)k_1 + L_2 k_2 = k$

Differentiating this system or equations with respect to $t, k_1, k_2$ and $L_2$, after some
straightforward calculations, we get:

$$\frac{dL_2}{dt} = \frac{k_2 - k_1}{D}$$

with

$$D = M'(L_2) - \frac{(k_1 - k_2)[f''(k_1) - \theta f''(k_2)]}{L_1 \theta f''(k_2) + L_2 f''(k_1)} t - \frac{(k_2 - k_1)^2 f''(k_1) \theta f''(k_2)}{L_1 \theta f''(k_2) + L_2 f''(k_1)}$$

Under the single crossing property, we have $f''(k_1) - \theta f''(k_2) < 0$, implying $D > 0$ and then

$$\frac{dL_2}{dt} > 0$$

The higher the tax rate the central planner has to charge on capital, the higher the
population of region 2, the larger the disequilibrium between regions.

**Appendix 5**

Let us now consider the case of an upward limit on the labor tax, $\tau_1 \leq \bar{\tau} < z$ and $\tau_2 \leq \bar{\tau} < z$, with $u'(z)$, so that the planner is no longer able to rest upon households
for financing the production of the public good. Now, the planner is maximizing $W$
under the following constraints:

$c_1 + z_1 = f(k_1) + (k - k_1)(f'(k_1) - t_1)$

$c_2 + z_2 = \theta f(k_2) + (k - k_2)(\theta f'(k_2) - t_2)$

$f'(k_1) - t_1 = \theta f'(k_2) - t_2$

$M(L_2) = c_2 + u(z_2) - c_1 - u(z_1)$

$L_1 k_1 + L_2 k_2 = k$

$z_1 \leq \bar{\tau} + k_1 \tau_1$

$z_2 \leq \bar{\tau} + k_2 \tau_2$

hence the Lagrangian:

$\gamma = L_1[c_1 + u(z_1)] + L_2[c_2 + u(z_2)] - \Omega(L_2) - \lambda_1[c_1 + z_1 - f(k_1) - (k - k_1)(f'(k_1) - t_1)] - \lambda_2[c_2 + z_2 - \theta f'(k_2) - (k - k_2)(\theta f'(k_2) - t_2)] - \mu[L_1 k_1 + L_2 k_2] - \zeta[f'(k_1) - t_1 - \theta f'(k_2) + t_2] - \xi[c_2 + u(z_2) - c_1 - u(z_1) - M(L_2)] - \eta_1(z_1 - \bar{\tau} - k_1 \tau_1) - \eta_2(z_2 - \bar{\tau} - k_2 \tau_2)$
Differentiating the Lagrangian with respect to \( z_1, z_2, c_1, c_2, k_1, k_2, t_1, t_2 \) and \( L_2 \), and rearranging, we get the following first order conditions for a second best outcome:
\[
\begin{align*}
\xi &= L_2 - \lambda_2 \\
\xi &= \lambda_1 - L_1 \\
(u'(z_1) - 1)\lambda_1 &= \eta_1 \\
(u'(z_2) - 1)\lambda_2 &= \eta_2 \\
\zeta &= \lambda_1 k - (\lambda_1 + \eta_1)k_1 \\
\zeta &= (\eta_2 + \lambda_2)k_2 - \lambda_2 k \\
\mu L_1 &= \lambda_1 u'(z_1)t_1 + \eta_1 k_1 f''(k_1) \\
\mu L_2 &= \lambda_2 u'(z_2)t_2 + \eta_2 k_2 \theta f''(k_2) \\
M(L_2) &= [c_2 + u(z_2)] - [c_1 + u(z_1)] \\
\xi M'(L_2) &= \mu (k_2 - k_1) \\
c_1 + z_1 &= f(k_1) + (k - k_1)(f'(k_1) - t_1) \\
c_2 + z_2 &= \theta f(k_2) + (k - k_2)(\theta f'(k_2) - t_2) \\
f'(k_1) - t_1 &= \theta f'(k_2) - t_2 \\
z_1 &= \bar{t} + k_1 t_1 \\
z_2 &= \bar{t} + k_2 t_2
\end{align*}
\]
Note that, now,
\[
\begin{align*}
u'(z_1) &= 1 + \frac{\eta_1}{\lambda_1} > 1 \\
u'(z_2) &= 1 + \frac{\eta_2}{\lambda_2} > 1
\end{align*}
\]
and then the planner is no longer following the standard Samuelson rule; compared to the first best, he is underproviding the public good in both regions.

Differentiating this system with respect to all the unknowns and after some tedious calculations, we get the following derivatives
\[
\begin{align*}
\frac{dt_1}{d\bar{t}} &= \frac{F_1}{F} < 0 \\
\frac{dt_2}{d\bar{t}} &= \frac{F_2}{F} < 0 \\
\frac{dt_1}{d\bar{t}} - \frac{dt_2}{d\bar{t}} &= \frac{F_1 - F_2}{F} < 0 \\
\frac{dk_1}{d\bar{t}} &= \frac{E_1}{D_1 F} > 0 \\
\frac{dk_2}{d\bar{t}} &= \frac{E_2}{D_2 F} \\
\frac{dz_1}{d\bar{t}} &= \frac{F + k_1 F_1}{F} > 0 \\
\frac{dz_2}{d\bar{t}} &= \frac{F + k_2 F_2}{F}
\end{align*}
\]
\[
\frac{dL_2}{d\bar{r}} = -\frac{1}{k_2 - k_1} \left( \frac{L_1 E_1}{D_1 F} + \frac{L_2 E_2}{D_2 F} \right)
\]

With:

\[A_1 = M'(L_2)k_1 L_1 u''(z_1) < 0\]
\[A_2 = M'(L_2)k_2 L_2 u''(z_2) < 0\]
\[B_1 = (k_2 - k_1)^2 k_1 f''(k_1) u''(z_1) > 0\]
\[B_2 = (k_2 - k_1)^2 k_2 f''(k_2) u''(z_2) > 0\]
\[D_1 = -(k_2 - k_1)^2 f''(k_1) f''(k_2) + L_2 M'(L_2)f''(k_1) + L_1 M'(L_2)f''(k_2) < 0\]
\[D_2 = -(k_2 - k_1)^2 f''(k_1) f''(k_2) + L_2 M'(L_2)f''(k_1) + L_1 M'(L_2)f''(k_2) < 0\]
\[F = [k_1 k_2 (B_1 B_2 - A_1 B_2 - A_2 B_1) - (k_1 (A_1 - B_1) + k_2 (A_2 - B_2))(k_2 - k_1)^2 + (k_2 - k_1)^4] > 0\]
\[F_1 = k_2 (A_1 B_2 + A_2 B_1 - B_1 B_2) - (k_2 - k_1)^2 (B_1 - A_1 - A_2) < 0\]
\[F_2 = k_1 (A_1 B_2 + A_2 B_1 - B_1 B_2) - (k_2 - k_1)^2 (B_2 - A_1 - A_2) < 0\]
\[F_1 - F_2 < 0\]
\[E_1 = L_2 M'(L_2)(F_1 - F_2) - (k_2 - k_1)^2 \theta f''(k_2) F_1 < 0\]
\[E_2 = L_1 M'(L_2)(F_1 - F_2) - (k_2 - k_1)^2 f''(k_1) F_2\]

**Appendix 6**

We will work in two stages. First, we look at the optimal choice of \(z_1, z_2, c_1, c_2, k_1, k_2\) for \(L_2\) given, hence population-dependent welfare function \(W(L_2)\). Then, we look at the first and second order conditions for the maximization of \(W(L_2)\).

The problem is similar to Appendix 1. \(L_2\) being given, the planner maximizes \(W\) with respect to \(z_1, z_2, c_1, c_2, k_1, k_2\), under the following constraints:

\[L_1 \theta_1 f(k_1) + L_2 \theta_2 f(k_2) = L_1 (c_1 + z_1) + L_2 (c_2 + z_2)\]
\[L_1 k_1 + L_2 k_2 = k\]

hence the Lagrangian:

\[Y_8 = L_1 (c_1 + u(z_1)) + L_2 (c_2 + u(z_2)) - \Omega(L_2) + \lambda [L_1 \theta_1 f(k_1) + L_2 \theta_2 f(k_2) - L_1 (c_1 + z_1) - L_2 (c_2 + z_2)] - \xi [L_1 k_1 + L_2 k_2 - k]\]

Differentiating the Lagrangian with respect to \(z_1, z_2, c_1, c_2, k_1, k_2\), and rearranging, we get the following first order conditions:

\[\theta_1 f'(k_1) = \theta_2 f'(k_2)\]
\[u'(z_1) = u'(z_2) = 1\]

We again find the equality of marginal productivities and the Samuelson rule, the later implying \(z_2 = z_1 = z\) with \(u'(z) = 1\).

The solution to this problem leads to a solution correspondence, \(W(L_2)\). Then, we maximize \(W(L_2)\). Knowing that \(\theta_1 = \theta(L_1) = \theta(1 - L_2)\) and \(\theta_2 = \theta(L_2)\) and using the envelope theorem, we get the first order condition:

\[\frac{\partial w(L_2)}{\partial L_2} = \frac{\partial r_a}{\partial L_2} = c_2 + u(z_2) - c_1 - u(z_1) + \lambda [L_2 \theta'(L_2)f(k_2) + \theta(L_2)f(k_2) - c_2 - z_2 - L_1 f(k_1) \theta'(L_1) - \theta(L_1)f(k_1) + c_1 + z_1] - \xi (k_2 - k_1) - M(L_2) = 0\]
which simplifies to:
\[
M(L_2) = \theta_2[f(k_2) - k_2f'(k_2)] - \theta_1[f(k_1) - k_1f'(k_1)] + [L_2 \theta'_2 f(k_2) - L_1 \theta'_1 f(k_1)]
\]
Where \( \theta'_1 = \theta'(L_1) \) and \( \theta'_2 = \theta'(L_2) \)
Let us note that the symmetric outcome, \( L_1 = L_2 = 0.5 \), which implies \( \theta_1 = \theta_2 = \theta(0.5), \theta'_1 = \theta'_2 = \theta'(0.5) \) and \( k_1 = k_2 = k \), obviously meets this first order condition.

Because the function \( \theta(L) \) is not concave, we have to beware of the second order condition. Differentiating \( \frac{\partial W(L_2)}{\partial L_2} \) with respect to \( L_2 \), we get:
\[
\begin{align*}
\frac{\partial^2 W}{\partial L_2^2} &= \frac{\partial^2 W}{\partial L_2^2} = \theta'_2 f(k_2) - k_2 \theta'_2 f'(k_2) + L_2 \theta''_2 f(k_2) + 2 \theta'_1 f(k_1) - k_1 \theta'_1 f'(k_1) + \\
&+ L_1 \theta''_1 f(k_1) - M'(L_2) + \\
&\quad [L_2 \theta'_2 f'(k_2) - \theta_2 k_2 f''(k_2)] \frac{\theta_2 f''(k_2) - k_2 f''(k_2) + L_2 \theta''_2 f(k_2) + \theta_1 \theta'_1 f'(k_1)]}{L_2 \theta_1 f''(k_1) + L_1 \theta_2 f''(k_2)} \\
&\quad + \left[ \theta_1 f''(k_1) - \theta_1 \theta'_1 f'(k_1) \right] \frac{\theta_1 f''(k_1) - k_2 f''(k_2) + L_2 \theta''_2 f(k_2) + \theta_1 \theta'_1 f'(k_1)}{L_2 \theta_1 f''(k_1) + L_1 \theta_2 f''(k_2)}
\end{align*}
\]
At the symmetric outcome, \( L_1 = L_2 = 0.5, \theta_1 = \theta_2 = \theta(0.5), \theta'_1 = \theta'_2 = \theta'(0.5) \) and \( k_1 = k_2 = k \), and assuming that \( M'(0.5) = 0 \), this derivative simplifies to:
\[
\frac{\partial^2 W}{\partial L_2^2} = 4 \theta f(k) + \theta'' f(k) + \left( \frac{\theta f''(k)}{\theta'' f(k)} - 4k \right) \theta f'(k)
\]
Where \( \theta = \theta(0.5), \theta' = \theta'(0.5) \) and \( \theta'' = \theta''(0.5) \). This expression may also be written as
\[
\frac{\partial^2 W}{\partial L_2^2} = \theta f(k) \left[ 4 + \theta'' + \left( \frac{\theta f''(k)}{\theta'' f(k)} - 4k \right) \frac{f'(k)}{f(k)} \right] \\
= \theta f(k) \left[ 4 + 2A + \frac{\eta^2}{2\eta^2 - 4\eta} \right] = \frac{2\theta f(k)}{A} \left[ A^2 + 2(1 - \eta)A + \frac{\eta^2}{4k^2} \right]
\]
where \( A = L_i \frac{\theta''}{\theta'} = 0.5 \frac{\theta''}{\theta'} \) is the elasticity with respect to population of the marginal increase in the agglomeration externality and \( \eta = \frac{k f'(k)}{f(k)} \) is the elasticity of production function with respect to capital.
The ratio \( \frac{2\theta'f(k)}{A} \) being positive, \( \frac{\partial^2 W}{\partial L_2^2} \) has the same sign as the quadratic polynomial \( A^2 + 2(1 - \eta)A + \frac{\eta^2}{4k^2} \). The discriminant of this polynomial is

\[
\Delta^2 = 4(\eta - 1)^2 - \frac{\eta^2}{k^2} = \left[ (2 + \frac{1}{k})\eta - 2 \right] \left[ (2 - \frac{1}{k})\eta - 2 \right]
\]

Two cases must be distinguished:

- When \( \Delta^2 < 0 \Leftrightarrow \eta \in \left[ \frac{2k}{2k+1}, \frac{2k}{2k-1} \right], \frac{\partial^2 W}{\partial L_2^2} > 0 \) whatever \( A \).

- When \( \Delta^2 > 0 \Leftrightarrow \eta \notin \left[ \frac{2k}{2k+1}, \frac{2k}{2k-1} \right], \frac{\partial^2 W}{\partial L_2^2} \leq 0 \) for \( A \in [A_1, A_2] \) and \( \frac{\partial^2 W}{\partial L_2^2} > 0 \) for \( A \notin [A_1, A_2] \), with \( A_1 = 2(\eta - 1) - \Delta \) and \( A_2 = 2(\eta - 1) + \Delta \)

Then, as soon as the elasticity with respect to population of the marginal increase in the agglomeration externality, \( A \), is high enough \( (A > A_2) \), we are sure that \( \frac{\partial^2 W}{\partial L_2^2} > 0 \) whatever \( \eta \), and then the symmetric outcome is a local minimum. The first best outcome is necessarily asymmetric.

Appendix 7

We look at the case of an asymmetric first best outcome; without loss of generality, we can assume that the largest (and most productive) region is region 2: \( L_2 > 0.5 \).

We know from Appendix 6 that, at the first best outcome, the following first order conditions must be met:

\[
\theta_1 f'(k_1) = \theta_2 f'(k_2) \tag{A7.1}
\]

\[
u'(z_1) = u'(z_2) = 1 \implies z_1 = z_2 = z \quad \text{with} \quad u'(z) = 1
\]

\[
M(L_2) = \theta_2 \left[ f(k_2) - k_2 f'(k_2) \right] - \theta_1 \left[ f(k_1) - k_1 f'(k_1) \right] + \left[ L_2 \theta'_2 f(k_2) - L_1 \theta'_1 f(k_1) \right] \tag{A7.2}
\]

When the planner charges taxes \( t_1, t_2, \tau_1 \) and \( \tau_2 \), as in Appendix 2, at equilibrium, the following conditions must be met:

\[
\rho = f'(k_1) - t_1 = \theta f'(k_2) - t_2 \tag{A7.3}
\]

\[
M(L_2) = c_2 + u(z_2) - c_1 - u(z_1) \tag{A7.4}
\]

For the equilibrium to implement the first best optimum, (A7.1), (A7.2), (A7.3) and (A7.4) must hold simultaneously. (A7.1) and (A7.3) jointly imply \( t_1 = t_2 \). Knowing that at the first best optimum, \( z_1 = z_2 = z \), and using the expressions for consumption, (A7.4) becomes:

\[
M(L_2) = c_2 - c_1 = \theta_2 \left[ f(k_2) - k_2 f'(k_2) \right] - \theta_1 \left[ f(k_1) - k_1 f'(k_1) \right] + \tau_1 - \tau_2
\]

And then, using (A7.2):

\[
\tau_1 - \tau_2 = L_2 \theta'_2 f(k_2) - L_1 \theta'_1 f(k_1)
\]

Then, a policy implementing the first best is characterized by:

\( z_1 = z_2 = z \) with \( u'(z) = 1 \)
\[ t_1 = t_2 = t \]
\[ \tau_1 - \tau_2 = L_2 \theta'_{2} f(k_2) - L_1 \theta'_{1} f(k_1) \]
\[ z = L_1 \tau_1 + L_2 \tau_2 + kt \]
Literature


