A stochastic model of sovereign credit spread

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Abstract

In this paper, we investigate the determinant of Sovereign Credit Spread from a stochastic model for four emerging countries in the 2000-2011 period. This depends on two default policies that we propose: an increase in corporate income tax and partial reduction of debt. We opt for the maximum likelihood estimation provided by Duan (1994) to find the sovereign credit spread, and cointegration estimation in order to validate the model. Our findings can be summarized as follow: i, the evolution of credit spread is fairly homogeneous compared with a benchmark credit spread; ii, the sign and Adjusted R-square value explain a strong relationship between the estimated sovereign credit spread and the benchmark; iii, the existence of cointegration in long-run between the estimated sovereign credit risk and the benchmark.

Key words: Sovereign credit spread, maximum likelihood, cointegration test
JEL classification: F34, G12, C22, C32

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1 Introduction and Literatures

The economic crisis emerged on a large scale in the emerging countries. The beginning of the Mexican tequila crisis (1994), the Asian crisis that spread from Thailand (1997) and the Russian debt crisis (1998) are due to political crises and bad executive policies. Especially in the early 2000s, the failure of regulation and risk management lead to crises in the emerging countries; the currency devaluation in Turkey (2000) and mostly in Latin American countries (Argentina’s default in 2001, Brazil’s crisis in 2002...). More recently, the global crisis system derives from the U.S subprime mortgage crisis in 2007 which influences strongly other countries.

Since then, measuring sovereign risk has been a very important task for policy-makers and regulators. Many indicators of country risk have been used recently such as sovereign credit ratings by agencies, default probability, sovereign credit spread, Credit Default Swap Spread (CDS spread)...

We begin to review the existing various literatures of sovereign default: the structural model, the econometric model and the stochastic model. The structural model of sovereign default was developed from the corporate credit risk model (Merton, 1974). It is based on the option pricing theory of Black and Scholes (1973) and the national balance sheet. Gray et al. (2007); François et al. (2011) consider a sovereign equity as a call option, which defines the sum of domestic currency debt plus the monetary base. Besides, Karmann and Maltritz (2009) focus on how to calculate directly sovereign’s ability-to-pay as the sum of all foreign exchange reserves and the discounted steady state capital flow of balance trade.

From the econometric model, the country risk is determined by the fundamental macroeconomic variables. In order to estimate the country risk, we can summarize some explanatory variables: Cantor and Packer (1996) find six variables affecting the sovereign credit rating which are per capita income, GDP growth, inflation rate, external debt, default history and an economic development indicator. Baek et al. (2005) show the link between sovereign risk in emerging countries and macroeconomic variables such as government budget balance and current account balance. Additionally, Mellios and Paget-Blanc (2006) determine two other factors that are government income and change in the real exchange rate which have a positive impact on the default probability. Georgievska et al. (2008) realize a grand empirical study on a panel of 124 emerging countries during the 1981-2002 period. They suggest three principal variables to explain the country risk: the total debt to GDP ratio and Export to GDP ratio represent solvency variables, international reserves to GDP ratio expresses liquidity and currency account balance to GDP ratio and imports to GDP ratio variables represent macroeconomic variables.

In the paper, we focus on the sovereign credit spread, which is the differential between yields on risky debt and those on what might be considered risk-free government bonds (Remolona et al. 2007). It is presented in the dynamic stochastic equilibrium model. Eaton and Gersovitz (1981) show that the government chooses to repay its debt because the impact of default on reputation will degrade the access to credit on the international market. Agreeing with it, Bulow and Rogoff (1989) study the exclusion from the international market due to default reputation. In reality, although many emerging countries were downgraded and defaulted on their debt, foreign investors have come back after the government revived the economy and stabilised economic growth, eg: Argentina, Mexico, Russia, Malaysia, Ecuador. Indeed, it is said that two typical examples were when Malaysia (1997) and Ecuador (2008) defaulted, all investors thought that these two countries had lost access to credit international market, but when they restored economic stability,
international investors came back again. Thus, the importance for investors is the economic outlook and macroeconomic stability, the reputation default being a very easy thing to forget. 

Yue (2010) studies the role of renegotiation when the sovereign defaults, and presents the bargaining power in reducing debt if the government does not have the capacity to repay. In addition, Andrade (2009) focuses on the asset pricing and renegotiation of sovereign debt when a country has negative economic growth and a bad endowment shock. Agreeing with the choice to default, the government accepts to pay the default cost. The default cost is mentioned in the work of Borensztein and Panizza (2008); Arellano (2008); Andrade and Chhaochharia (2011). There are some default models of reserve optimisation like Ben-Bassat and Gottlieb (1992); Alfaro and Kanczuk (2009). The interaction between fiscal policy and sovereign risk appears in Cuadra et al. (2010); Hatchondo et al. (2012). In the paper of Andrade (2009), the author creates a yield spread on sovereign bond by using two ratios of Price-Earnings (P/E) and expected return. His model suggests that the P/E ratio of an emerging market stock decreases with the average sovereign yield spread, and the valuation discount of stock price increases with the average sovereign yield spread. The empirical calibrates on Brazilian data from January 1998 to December 2007 based on EMBI+ Index, that represents a weight portfolio. Besides, Jeanneret (2013) studies the dynamic sovereign credit risk model by finding the sovereign credit spread. His model indicates that when the sovereign defaults, the government will be incited to issue debt and to negotiate with its lenders to reduce its debt. The empirical results of this paper for the period 2000-2011 derive from two groups: a first group of emerging markets, representing Brazil, Columbia, Mexico, Peru, Russia and Turkey; a second group of European countries, representing France, Greece, Ireland, Italy, Portugal and Spain.

In this paper, we modify and extend the dynamic sovereign default model of Jeanneret (2013). In our model, we propose two parallel policies if the sovereign defaults: the government increases corporate income tax and negotiates to reduce its debt. The principal contribution of this paper is a method to determine the sovereign credit spread following the two policies and to validate the model. The empirical estimations apply for four emerging countries: Brazil, Mexico, Peru and Turkey in the period 2000-2011.

This paper is organized as follows. In the second section, we present the model. In the next section, we show the empirical results. The last section is the conclusion.

2 Model

We suppose that a country consists of a government and a representative firm. The government has no international reserves. The firm asset, $V_t$, is represented by its income and follows a Geometric Brownian Motion (GBM) with a drift $\mu$ and volatility $\sigma$:

$$dV_t = \mu(1 - \tau - \Delta)V_t + \sigma(1 - \tau - \Delta)V_t dW$$

where $W$ is a Brownian motion, $\Delta \geq 0$.

This government taxes on the firm’s income at a constant rate $\tau$. Thus, the government’s fiscal revenue at the time $t$ is $\tau V_t$, and the net firm’s asset is $(1 - \tau)V_t$. The government pays a perpetual debt service $C$ to its lenders.

The government defaults when sovereign asset $V_t$ reaches a threshold default $V_D$, called barrier
default, at time $T_D = \inf \{ t \geq 0, V_t \leq V_D \}$\footnote{We can write $\tau V_t < V_D$, so $V_t < V_D$ where $V_D = V_D1/\tau$.}. If the government defaults, we propose two parallel policies: First, the government will negotiate with its lender to reduce the debt service by a fraction $\phi \in [0, 1]$. Second, the government will increase firm’s income rate from $\tau$ to $\tau + \Delta$ ($\Delta > 0$). In the absence of government default, $\Delta = 0$. According to Shiller (2013), "increasing taxes during an economic crisis makes perfect sense". In addition, in the European crisis 2011, three countries Greece, Spain, Portugal that increase the income tax to decrease the deficit (OCDE-Publishing, 2013). It is the idea that we compose in this model.

If the government chooses to default, it must pay default cost\footnote{The asset value at the default point $T_D$ is $V_{TD}$ that equals to default threshold $V_D$.}:

$$\lambda E_t \left[ \tau V_{TD} e^{-r(T_{TD} - t)} \right] \quad (2)$$

where $\lambda$ is a fraction of the reduction the firm asset, $r$ is risk-free interest rate. Jeanneret (2013) argues the sovereign asset is a fraction of the firm asset because the firm pays its income to the government. At the default time $T_D$, in order to pay a default cost, it must reduce a fraction of firm asset $\lambda$. The equation (2) is the present value of the default cost.

**Proposition 1.** We present the "net sovereign asset (NSA)" composed the present value of the net government’s income minus default cost:

$$NSA = E_t \left[ \int_t^{T_D} (\tau V_u - C) e^{-r(u-t)} du + \int_{T_D}^{\infty} \{ (\tau + \Delta) V_u - (1 - \phi)C \} e^{-r(u-t)} du \right] - \lambda E_t \left[ \tau V_{TD} e^{-r(T_{TD} - t)} \right]$$

Equation (3) shows: the first term is the present value of government’s fiscal revenue minus coupon until the default time $T_D$. The second term is the present value of government’s fiscal revenue minus coupon after default, and applied the parallel policies. At the default time $T_D$, the government increases the income tax $\tau + \Delta$, parallel with that policy, the government negotiates to reduce debt service a fraction $\phi$. The rest of coupon service is $(1 - \phi)C$. The third term is the default cost.

By using Lemma A.1 and Lemma A.2, we obtain the net sovereign asset:

$$NSA = \frac{\tau V_t}{r - \mu} - \frac{C}{r} \left( \frac{V_t}{V_D} \right)^{\beta} \left[ \left( \frac{\Delta}{r - \mu} - \lambda \tau \right) V_D + \frac{\phi C}{r} \right]$$

where $\beta = -\frac{2r}{\sigma^2}$

**Proof.** See Appendix A.

The government policy is to maximize the net sovereign asset value. The optimal default barrier is found from this policy displayed in the Proposition 2 below:

**Proposition 2.** The default barrier, $V_D$, is determined at the time when the sovereign asset infers to the default barrier:

$$V_D^* = \frac{C \phi \beta (r - \mu)}{r (1 - \beta)[\Delta - \lambda \tau (r - \mu)]} = C \phi$$

(5)
where $\varphi = \frac{\phi \beta (r - \mu)}{r (1 - \beta) [\Delta - \lambda \tau (r - \mu)]}$; $\beta = -\frac{2r}{\sigma^2}$

**Proof.** See Appendix A.

The central purpose of this paper is how to determine the sovereign credit spread from the two policies: increase corporate income tax and reduce a part of the debt. The sovereign credit spread is the difference between the yield on risky debt and risk-free rate, which is found by the equation (6) in the Proposition 3.

**Proposition 3:** Sovereign Credit Spread (SCS) is measured by:

$$SCS = r \left[ \frac{1}{1 - \phi \left( \frac{V_t}{V} \right)^{\beta}} - 1 \right]$$  \hspace{1cm} (6)

**Proof.** See Appendix A.

**Maximum Likelihood Estimation method**

In order to estimate the sovereign credit spread in the equation (6), it must find the unknown asset value that cannot be observed. To do that, we study here the Maximum Likelihood Estimation (MLE) proposed by Duan (1994). The transformed-data method will find unknown asset value from the observed equity value through the log-likelihood function.

We have the observed equity is a function of unknown asset value $E = f(V)$, inferring $V = f^{-1}(E)$. We express the log-likelihood function for the observed equity.

$$L(E, \theta) = L(V, \theta) - \sum_{t=2}^{n} \ln \frac{\partial f(V_t)}{\partial V_t} = L(V, \theta) - \sum_{t=2}^{n} \ln \frac{\partial E_t}{\partial V_t}$$  \hspace{1cm} (7)

where $\theta = (\mu, \sigma)$, $L(V, \theta)$ is the log-likelihood function for unknown asset $V_t$.

We use $\partial E_t/\partial V_t = N(d_t)$, we obtain:

$$L(E, \theta) = L(V, \theta) - \sum_{t=2}^{n} \ln(N(d_t))$$  \hspace{1cm} (8)

$$L(E, \theta) = -\frac{n-1}{2} \ln(2\pi) - \frac{n-1}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=2}^{n} [\ln \left( \frac{V_t}{V_{t-1}} \right) - \mu]^2 - \sum_{t=2}^{n} \ln(N(d_t))$$  \hspace{1cm} (9)

Duan (1994) uses the maximum method of quadratic Hill-Climbing proposed by Goldfeld et al. (1966) to find the maximum likelihood value. In our empirical results, we apply the *simplex algorithm* to maximum likelihood proposed by Lagarias et al. (1998). Using maximum likelihood estimation to find estimated value $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ and $V_t$ from input data of observed equity and default barrier.

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\[3\] To find the maximum likelihood value $L(E)$, we switch the problem by finding the minimum value $-L(E)$
3 Empirical results

3.1 Data Description

In this paper, we use daily data from the period 2000-2011 for four emerging countries: Brazil, Mexico, Peru and Turkey. MSCI is a stock market price which represents the sovereign equity index. EMBI is the Emerging Market Bond Index which represents the benchmark sovereign credit spread. Both MSCI and EMBI are taken from Datastream.

To set-up the remaining variables, we use the calibrate value from Jeanneret (2013) because we calculate in the same period and the same country. We set: the average 10-year U.S. Treasury rate represents the risk-free interest rate $r = 0.04$, the corporate income tax for emerging country $\tau = 0.3$, the default cost rate $\lambda = 0.05$, reduced rate of the debt $\phi = 0.6$ and $C = 1\%, \Delta = 1\%$.

3.2 Estimation from MLE

The difference of default threshold of each country depends on the estimated value of $\hat{\mu}, \hat{\sigma}$ from the maximum likelihood estimation. The loop of simplex algorithm in MLE requires a starting point of $\mu, \sigma$. Like Jeanneret (2013), we use the mean and volatility of MSCI growth represent for starting point of $\mu, \sigma$. The estimated coefficients $\hat{\mu}, \hat{\sigma}$ are reported in Figure 1 in below:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Brazil</th>
<th>Mexico</th>
<th>Peru</th>
<th>Turkey</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.3014</td>
<td>0.2511</td>
<td>0.2575</td>
<td>0.4550</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>0.1315</td>
<td>0.1152</td>
<td>0.1761</td>
<td>0.0933</td>
</tr>
</tbody>
</table>

Source: author’s calculation

The estimated default barrier in equation (5) is calculated from estimated coefficient of $\hat{\mu}, \hat{\sigma}$.

3.3 Comparing between estimated Credit Spread and benchmark Credit Spread

The estimated sovereign credit spread is calculated from the estimated coefficient of $\hat{\mu}, \hat{\sigma}$ in Figure 1 and the estimated asset value from maximum likelihood estimation.

We compare the estimated sovereign credit spread with the Emerging Market Bond Index represented the benchmark sovereign credit spread in order to verify the tendency and the fluctuation of the two curves. The comparison shows in Figure 2 below:
As shown in Figure 2, the estimated sovereign credit spread and EMBI are fairly homogeneous and have a similar tendency over the period. In the period 2000-2011, there are two crises, in 2002 and the global financial in 2007. The estimated country risk as well as the EMBI benchmark illustrates an increase in these two crises periods. In the case of Brazil and Mexico, the two curves coincide almost over the whole period. Moreover, there is a small residual in 2002 for Brazil and after 2007 for Mexico. In the case of Peru and Turkey, the two curves are not coincidental, but more importantly they have the same tendency.

For that reason, we will introduce the econometric model in order to highlight the evolution of the two curves that are significant over the period, and especially if in the long-run the cointegration of the two series exists. This question is answered in the next section.

Source: author’s calculation.
Notes: (a), (b), (c) and (d) are the curve of Brazil, Mexico, Peru and Turkey respectively. Both the curves of estimated sovereign credit spread and EMBI are normalised to unity in 2000.
3.4 Model Validation

In this section, we investigate two econometric methods to validate the sovereign credit spread. We test the evolution between estimated sovereign spread and the benchmark spread. If two indices have the same evolution, we will apply the second econometric method to verify the long-run existence of the evolution by using cointegration test.

3.4.1 Evolution between estimated sovereign spread and benchmark spread

The graphics in Figure 2 show the same evolution of two curves of estimated sovereign spread and the benchmark spread. Here, we test the relationship of evolution of two curves. We use this equation below:

\[ SCS_t = \alpha + \gamma EMBI_t + \varepsilon_t \]  

where \( SCS \) is the estimated sovereign credit spread from the model, EMBI is Emerging Market Bond Index, represented for country’s observed credit spread. The expected sign of \( \gamma \) is positive.

The econometric result is mentioned in Figure 3:

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Brazil</th>
<th>Mexico</th>
<th>Peru</th>
<th>Turkey</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.231***</td>
<td>-0.0589***</td>
<td>-0.0837***</td>
<td>0.736***</td>
</tr>
<tr>
<td>(0.0053)</td>
<td>(0.0150)</td>
<td>(0.0061)</td>
<td>(0.0051)</td>
<td></td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.634***</td>
<td>0.8158***</td>
<td>0.7549***</td>
<td>0.437***</td>
</tr>
<tr>
<td>(0.005)</td>
<td>(0.0163)</td>
<td>(0.00693)</td>
<td>(0.0044)</td>
<td></td>
</tr>
<tr>
<td>Adj R-squared</td>
<td>0.835</td>
<td>0.443</td>
<td>0.791</td>
<td>0.756</td>
</tr>
<tr>
<td>Observations</td>
<td>3130</td>
<td>3130</td>
<td>3130</td>
<td>3130</td>
</tr>
</tbody>
</table>

Notes: standard errors in parentheses; *** means that the coefficient is significant at the 1% level.

The estimated coefficients are reported in Figure 3. The coefficient of \( \gamma \) is positive and significant at 1% level as expected which are 0.634, 0.8158, 0.7549, 0.437 for four countries Brazil, Mexico, Peru and Turkey respectively. Furthermore, the Adjusted R-squared values of the four countries are high which clarifies a strong relationship between the SCS and the EMBI index. This result reflects the evolution of estimated sovereign credit risk of the model explaining by the evolution of EMBI in the period 2000-2011.

3.4.2 Cointegration test between estimated sovereign spread and benchmark spread

In the previous section, we obtain the same evolution of SCS and EMBI. In this section, we propose the cointegration test between estimated sovereign spread and EMBI in the long-run. If the existence of cointegration between two series, we can validate the calculated model of sovereign credit spread.
Before testing the cointegration test, we start by testing the stationary of two series SCS and EMBI. For each test, the optimal lag is obtained by choosing the critical AIC selection and Augmented Dickey-Fuller (ADF) statistic. The null hypothesis is that the series contains a unit root. We study two models: the first model with intercepts only, the second model with intercepts and trend. The results are mentioned in Figure 4 below:

### Figure 4: Unit root test

<table>
<thead>
<tr>
<th>Model</th>
<th>Brazil</th>
<th>Mexico</th>
<th>Peru</th>
<th>Turkey</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stat-SCS Constant</td>
<td>-0.7359</td>
<td>-1.0074</td>
<td>-0.7190</td>
<td>-1.7260</td>
</tr>
<tr>
<td></td>
<td>(0.8358)</td>
<td>(0.7527)</td>
<td>(0.8401)</td>
<td>(0.4180)</td>
</tr>
<tr>
<td>Stat-SCS Constant and Trend</td>
<td>-2.0966</td>
<td>-2.1620</td>
<td>-1.7341</td>
<td>-2.8074</td>
</tr>
<tr>
<td></td>
<td>(0.5469)</td>
<td>(0.5101)</td>
<td>(0.7360)</td>
<td>(0.1946)</td>
</tr>
<tr>
<td>Stat-EMBI Constant</td>
<td>-1.6568</td>
<td>-2.3034</td>
<td>-1.7275</td>
<td>-1.9245</td>
</tr>
<tr>
<td></td>
<td>(0.4532)</td>
<td>(0.1710)</td>
<td>(0.4172)</td>
<td>(0.3212)</td>
</tr>
<tr>
<td></td>
<td>(0.3043)</td>
<td>(0.3227)</td>
<td>(0.2579)</td>
<td>(0.2501)</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Stat-∆EMBI Constant</td>
<td>-10.2390***</td>
<td>-10.0320***</td>
<td>-51.9538***</td>
<td>-18.5117***</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0001)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Stat-∆EMBI Constant and Trend</td>
<td>-10.2393***</td>
<td>-10.0295***</td>
<td>-51.9455***</td>
<td>-18.5090***</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>

Notes: ∆ is first difference, p-value in parentheses. *** means that we reject the null hypothesis of a unit root at the 1%.

Figure 4 gives the ADF statistics for SCS and EMBI series in level and in the first difference of each country. The two series SCS and EMBI in level are non-stationary based on p-value. The first difference rejects the null hypothesis of a unit root at the 1% level. Finally, all variables are integrated in the same order I(1). Therefore, we can use the cointegration test between SCS and EMBI in order to verify the long-run cointegration between SCS and EMBI.

### Cointegration test

In this section, we investigate the cointegration test to verify long-run cointegration between estimated sovereign credit spread and EMBI. The cointegration test for times series proposed by Granger (1981), Engle and Granger (1987) and Johansen (1991, 1995). According to Granger (1981), Engle and Granger (1987), if the residual in equation (10) is stationary, we can conclude the existence of cointegration between the two series SCS and EMBI. The disadvantage of the method of Engle and Granger (1987) is that it does not distinguish the multiple cointegration relationship. In this

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\(^4\text{Maxlag}=28\)
section, we apply the cointegration test of [Johansen 1991, 1995] in order to cite also the number of cointegration. We test the short-run dynamic model from equation (10):

\[ \Delta SCS_t = \beta \Delta EMBI_t + \delta e_{t-1} + \nu_t \]  

(11)

where \( \delta < 0 \), \( e_t = SCS_t - \hat{\gamma}EMBI_t - \hat{\alpha} \).

The null hypothesis is no-cointegration and the alternative hypothesis is cointegration between the two series SCS and EMBI. The results report in Figure 5:

![Figure 5: Cointegration test: Trace & Maximum Eigenvalue statistics](image)

The results observed in Figure 5 conclude that both Trace test and Max-Eigenvalue test indicate the existence of one cointegration in long-run between estimated sovereign credit spread and EMBI for Brazil, Mexico, and Peru. This result also shows that there are two cointegrations in long-run for Turkey.

Therefore, the evolution between the estimated sovereign spread and EMBI is homogeneous and significant in long-run.

4 Conclusion

This paper provides a stochastic model to determine the sovereign credit spread following two policies: an increase in corporate income tax and a reduction of their debt for four emerging countries. Our results for Brazil, Mexico, Peru, and Turkey indicate that two series, the estimated sovereign credit spread and EMBI have the same evolution. Furthermore, the innovation in this paper is using Johansen’s cointegration test (Johansen 1991, 1995) to explain the evolutionary significance of the sovereign credit spread. Compared with the work of Jeanneret (2013), our results explain that in long-run the evolution of sovereign credit spread is significant with that of the EMBI.

Interesting perspectives for future works would be: how to determine sovereign credit spread if a country has an initial gross endowment as in the paper of Andrade (2009) and the use of other collateral assets in order to borrow from the international market?
Acknowledgement

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Appendix

Lemma A.1. Let $V_t$ follows a GBM with the drift $\mu$ and volatility $\sigma$.

$$E_t \left[ \int_t^\infty e^{-r(u-t)}V_u du \right] = \frac{V_t}{\mu - r}$$

Proof. See Andrade (2009)

Lemma A.2. Let $V_t$ follows a GBM with the drift $\mu$ and volatility $\sigma$. Let $V_D < V_t$ and $T_D = \min \{ u : V_u = V_D \}$:

$$E_t \left[ e^{-r(T-t)} \right] = \left( \frac{V_t}{V_D} \right)^\beta$$

where $\beta = -\frac{2r}{\sigma^2}$


Proof of Proposition 1: The "net sovereign asset (NSA)" is proved based on the appendix of Andrade (2009):

$$NSA = E_t \left[ \int_t^{T_D} (\tau V_u - C) e^{-r(u-t)} du + \int_{T_D}^\infty \{ (\tau + \Delta) V_u - (1 - \phi)C \} e^{-r(u-t)} du \right] - E_t \left[ \lambda \tau V_D e^{-r(T_D-t)} \right]$$

where

$$E_t \left[ \int_t^{T_D} (\tau V_u - C) e^{-r(u-t)} du \right] = E_t \left[ \int_t^\infty (\tau V_u - C) e^{-r(u-t)} du \right] - E_t \left[ \int_{T_D}^\infty (\tau V_u - C) e^{-r(u-t)} du \right]$$

We use the chain property, we obtain:

$$\frac{\tau V_t}{r - \mu} - \frac{C}{r} - E_t \left[ e^{-r(T-t)} du \right] E_{T_D} \left[ \int_{T_D}^\infty (\tau V_u - C) e^{-r(u-t)} du \right] = \frac{\tau V_t}{r - \mu} - C - \left( \frac{V_t}{V_D} \right)^\beta \left( \frac{\tau V_D}{r - \mu} - C \right)$$

$$E_t \left[ \int_{T_D}^\infty \{ (\tau + \Delta) V_u - (1 - \phi)C \} e^{-r(u-t)} du \right] = \left( \frac{V_t}{V_D} \right)^\beta \left( \frac{(\tau + \Delta) V_D}{r - \mu} - \frac{(1 - \phi)C}{r} \right)$$

$$E_t \left[ \lambda \tau V_D e^{-r(T_D-t)} \right] = \lambda \tau V_D \left( \frac{V_t}{V_D} \right)^\beta$$

we obtain:

$$NSA = \frac{\tau V_t}{r - \mu} - \frac{C}{r} + \left( \frac{V_t}{V_D} \right)^\beta \left[ \left( \frac{\Delta}{r - \mu} - \lambda \tau \right) V_D + \frac{\phi C}{r} \right]$$
**Proof. of Proposition 2:** The "default barrier" is proved based on the appendix of [Jeanneret (2013)](Jeanneret2013). The default barrier is found by taking the first-order maximization of the net sovereign asset \( \frac{\partial NSA}{\partial V_t} \).

\[
\frac{\partial NSA}{\partial V_t} = \frac{\tau}{r - \mu} + \frac{\beta V_t^{\beta - 1}}{V_D^\beta} \left[ V_D \left( \frac{\Delta}{r - \mu} - \lambda \tau \right) + \frac{\phi C}{r} \right]
\]

Using the smooth-pasting condition \( \frac{\partial NSA(V=V_D)}{\partial V_D} = \frac{\partial NSA(V=V_D)}{\partial V_t} \) (Merton 1974)

where \( \frac{\partial NSA(V=V_D)}{\partial V_D} = \frac{\tau}{r - \mu} + \left( \frac{\Delta}{r - \mu} - \lambda \tau \right) \)

we obtain:

\[
V_D^* = \frac{C\phi \beta (r - \mu)}{r(1 - \beta) [\Delta - \lambda \tau (r - \mu)]} = C \varphi
\]

**Proof. of Proposition 3:**

Sovereign Credit Spread (SCS) is demonstrated based on [Jeanneret (2013)](Jeanneret2013)

\[
SCS = \frac{C}{D} - r = r \left[ \frac{1}{1 - \phi \left( \frac{V_t}{V_D} \right)^{\beta} - 1} \right] = r \left[ \frac{1}{1 - \phi \left( \frac{V_t}{C \varphi} \right)^{\beta} - 1} \right]
\]

where \( D \) is the sovereign debt. Using **Lemma A.2** we found:

\[
D = E_t \left[ \int_t^{T_D} C e^{-r(u-t)} du \right] + E_t \left[ \int_{T_D}^{\infty} (1 - \phi) C e^{-r(u-t)} du \right] = \frac{C}{r} \left[ 1 - \phi \left( \frac{V_t}{C \varphi} \right)^{\beta} \right]
\]
References


