Optimal Policy Rules Stabilize Bubbles

Jean-Bernard Chatelain and Kirsten Ralf

February 28, 2014

Abstract

This paper investigates identification and stability for two types of rational expectations equilibria in a dynamic Stackelberg game. Optimal policy feedback rules stabilizing asset prices and minimizing the volatility of interest rates are determined. For an interest rate rule with Central Bank commitment to financial stability, the optimal policy allows to stabilize asset price bubbles even for relatively small sensitivity (below 0.5) of the interest rate with respect to asset price deviations from their fundamental value in a financial accelerator model. Parameters in the interest rule do not face an identification problem. In the second type of equilibrium where the private sectors rules out bubbles according to Blanchard Kahn (1980) hypothesis, the sensitivity of the interest rate with respect to deviations of asset price from their fundamental value in the optimal policy rule is not identified. Any rule including asset prices is observationally equivalent to a rule which does not include asset prices but with a different weight on the capital gap.

JEL classification numbers: E44, E52, E58, O42.

*We thank for helpful comments Antoine d’Autume, Hippolyte d’Albis, Gunther Capelle-Blancard, Jean-Pierre Drugeon, Pedro Garcia Duarte, Roger Farmer, Stephane Gauthier, Mauro Napoletano, Michel de Vroey and Bertrand Wigniolle, as well as participants in the Macroeconomics in Perspective Workshop in Louvain la Neuve and the seminar "Dynamique de la macroéconomie" in Paris 1 Pantheon Sorbonne.

†Paris School of Economics, Université Paris I Pantheon Sorbonne, CES, Centre d’Economie de la Sorbonne, 106-112 Boulevard de l’Hôpital 75647 Paris Cedex 13. Email: jean-bernard.chatelain@univ-paris1.fr

‡ESCE Ecole Supérieure du Commerce Extérieur, 10 rue Sextius Michel, 75015 Paris, Email: Kirsten.Ralf@esce.fr.
1. Introduction

Does an optimal interest rule allow Central Banks with a financial stability mandate to stabilize asset prices and credit bubbles? Do these optimal rules allow to identify the sensitivity of the interest rate with respect to asset prices or credit in "augmented" Taylor rules when simulating and estimating the equilibrium of a macro-prudential dynamic stochastic general (DSGE) model or not? In other words, when the sensitivity of the interest rate with respect to asset price deviations is reported to be 0.5, is it exactly equivalent to a model where this sensitivity is 0 and the sensitivity of interest rate with respect to the output gap increased by a given factor or are these two scenarios different? This paper offers new insights on identification issues in "augmented" Taylor rules, as a complement to identification issues in "non augmented" Taylor rules highlighted by Cochrane (2011).

The current macro-prudential DSGE literature deals with augmented Taylor rules of the central bank including asset prices and other factors related to financial stability, such as households’, non-financial firms’ and banks’ leverage (e.g. Angeloni and Faia (2012)). This literature can be split into two main categories. Either bubbles are taken into account (e.g. Bernanke Gertler (2001)) or bubbles are not taken into account (e.g. Faia Monacelli (2007)) under the assumption that there are no bubbles on asset prices and on private sector leverage (no Ponzi game condition).

To this end, we study two types of equilibria with Central Bank optimal policy feedback rule under commitment to financial stability. This commitment may be justified by the recent emphasis on the financial stability mandate given to Central Banks. More precisely, the Central Bank participates as a Stackelberg leader in a linear quadratic rational expectations (LQ RE) dynamic game with the private sector (Ljungqvist and Sargent (2012, chapter 19), Woodford (2003b)). In type I equilibrium, bubbles can be stabilized by the Central Bank. In type II equilibrium, the private sector expectations rule out bubbles with the help of a no-bubble assumption.

In order to deal with the concern of Central Bankers that stabilizing asset prices and credit bubbles may lead to too much volatility of their interest rate,
we include interest smoothing in their cost function, see Woodford (2003b). In particular, we study a benchmark case where, despite the large relative weight of the cost of changing the interest rate in the loss function, the Central Bank is able to stabilize bubbles in a type I equilibrium.

The paper begins with the simplest example of a financial accelerator model where private debt is limited by the expected value of collateral which depends on asset price (Miller and Stiglitz (2010), Kiyotaki and Moore (1997)), including rational expectations diverging path for asset prices and private credit. This framework suggests that asset price bubbles are fueling credit bubbles. For the purpose at hand (identification issues on rules including asset price), we have already some interesting results, even though the Kiyotaki and Moore (1997) framework does not deal with hyperinflationary or deflationary diverging paths of consumer prices, along with asset prices and credit diverging paths.

Our results are that for type I equilibrium (stabilizing bubbles), identification suggests that an optimal rule including asset prices (and their related parameter) is equivalent to a history dependent optimal rule including a lagged value of the interest rate and the output gap. For type II equilibrium ("no-bubbles assumption in the private sector"), the parameter of asset price in an optimal rule cannot be identified. A rule including asset price (where for example, the sensitivity of the interest rate with respect to asset prices deviations is reported to be 0.5) is it exactly equivalent to a model where this sensitivity is 0 and the sensitivity of interest rate with respect to the output gap increased by a given factor. Not only type II sensitivities to asset prices are not identified, but their sensitivities reported in the current DSGE macro-prudential literature are also not markedly distinct from sensitivities in type I equilibrium. For example, in type I equilibrium, a sensitivity below 0.5 of the policy rate to deviations of asset price with respect to their long run fundamental value value is able to stabilize a range of rational asset price bubbles.

Finally, type II equilibrium is locally unstable (Burmeister (1980)): an infinitesimal error in their optimal rule by Central Bankers may lead to asset price bubbles despite the assumption of the ability of the private sector to rule them out. By contrast, type I equilibrium is bounded and locally stable, as in Woodford (2003b) model dealing with consumer prices. Simulating DSGE macro-prudential models with type I equilibrium with optimal policy under commitment to financial stability are particularly simple (see Ljungqvist and Sargent (2012), Giordani and Söderlind (2004)). In particular, as the final system is locally stable, bounded and determinate, its solution uses standard linear quadratic regulator numerical
analysis tools: all the numerical difficulties to obtain convergence related to finding a proper stable manifold are no longer a problem. It may be promising to use macro-prudential policy with commitment to financial stability more often than ad hoc augmented Taylor rules with non identified "augmented" parameters on asset prices and private sector leverage.

This paper develops a prototype model and proceeds as follows. Section 2 builds on Miller and Stiglitz (2010) and Kiyotaki and Moore (1997) financial accelerator model to show how two types of equilibrium may deal with asset price bubbles. Section 3 presents identification issues for type I and type II equilibria in the general case. Section 4 concludes with potential extensions.

2. A financial accelerator model

2.1. The private sector

This section presents the two sector model of Miller and Stiglitz (2010), Edison, Luangaram and Miller (2000), and Kiyotaki and Moore (1997). It is deliberately as simple as possible, and avoids dealing with hyperinflation or deflationary bubbles of consumer prices at the same time as asset price bubbles. Dividends are endogenous and we introduce a pre-determined variable besides the non-pre-determined asset price. The first sector consists of a continuum (with population size of 1) of identical, relatively impatient and infinitely lived borrowers. The second sector (indexed by i) consists of a continuum with population size of m identical, relatively patient and infinitely lived lenders. Both types of agents have a linear utility derived from consumption denoted c_i with discount factors denoted \( \beta \) (the index t is for time):

\[
U = \sum_{t=0}^{+\infty} \beta^t c_t \quad \text{and} \quad U' = \sum_{t=0}^{+\infty} \beta'^t c'_t \quad \text{with} \quad 0 < \beta < \beta' < 1.
\] (1)

A single output \( y \) is produced using only a capital input available in a fixed amount \( K \) and traded at an asset price \( q_t \), with \( k_t \) used in the borrowing sector and \( k'_i = \frac{K-k_t}{m} \) used in the lending sector. Borrowers use a constant return to scale technology, whereas lenders use a decreasing returns to scale technology parameterized as in Edison, Luangaram and Miller (2000):

\[
y_{t+1} = (A + c_0) k_t \quad \text{and} \quad y'_{t+1} = f(k'_i) = a \frac{K}{m} k'_i - \frac{1}{2} a k'_i^2
\] (2)
with $a > 0$, $A > 0$ and $c_0 > (\beta - 1)A > 0$: $c_0k_t$ is non tradeable production, which sets a minimal consumption level for lenders. Lenders marginal productivity is equal to zero when they own all the capital stock (borrowers’ capital equals zero: $k_t = 0$). In what follows, we set lenders population $m = 1$ without loss of generality:

$$\partial y_{t+1} / \partial k'_t = aK - a\left(\frac{K - k_t}{m}\right) = ak_t \text{ for } m = 1.$$ (3)

Borrowers’ flows of funds constraint states that value of their investment (depreciation of capital is set to zero) is equal to the variation of their debt plus cash flow from operating income (output less consumption less interest charges, with an interest rate denoted $r_t > 0$ and an interest rate factor denoted $R_t = 1 + r_t > 1$):

$$q_t (k_t - k_{t-1}) = (b_t - b_{t-1}) + Ak_{t-1} + c_0k_{t-1} - c_t - (R_t - 1)b_{t-1}.$$ (4)

Lenders limit their loans to borrowers such that the repayment of their debt and of its interest is backed by the expected value of borrowers’ capital used as collateral:

$$R_t b_t \leq E_t (q_{t+1}) k_t.$$ 

As long as $R_t < A$, in order to finance investment, optimizing borrowers wish to borrow the maximal amount allowed by lenders’ credit ceiling and to save as much as possible (they restrain their consumption to its minimal level: $c_t = c_0k_{t-1}$). Substituting the binding debt constraint on date $t$ and on date $t + 1$ in order to eliminate debt $b_t$ in the flow of funds equation, leads to this law of motion for capital owned by borrowers:

$$ak_t^2 = R_t Ak_t \Rightarrow k_{t+1} = f_1 (k_t, q_t, R_t) = \sqrt{\frac{R_tA}{a}} \sqrt{k_t}.$$ (5)

Lenders’ flow of funds equation is (with negative "debt" $b'_t < 0$):

$$q_t \left( k'_t - k'_{t-1} \right) = \left( b'_t - b'_{t-1} \right) + f(k'_t) - c'_t - (R_t - 1)b'_{t-1}.$$ 

Lenders’ behaviour is determined by an arbitrage equation between lending (with a rate of interest $r_t > 0$) versus investing in capital in their own firm, with marginal returns determined by the marginal productivity of capital in their own firm plus expected capital gains or losses over selling this capital next period:
\[ r_t = \frac{a k_t + E_t(q_{t+1}) - q_t}{q_t} \]
\[ \Rightarrow E_t(q_{t+1}) = f_2(k_t, q_t, R_t) = (1 + r_t) q_t + a k_t. \] (6)

The dynamical system consists of the law of motion of capital owned by lenders and of the above law of motion of expected asset prices. The long run equilibrium values are strictly positive with long run asset price determined by the perpetual rent formula. The interest rate is set by the Central Bank with an equilibrium value equal to lenders discount rate:

\[ k^* = \frac{R^* A}{a} \leq K, \quad q^* = \frac{R^* A}{R^* - 1} \quad \text{and} \quad R^* = 1/\beta' < A. \]

The log-linearized system around this equilibrium is (see appendix 1):

\[
\left( \begin{array}{c} (k_{t+1} - k^*) / k^* \\ (q_{t+1} - q^*) / q^* \\ \end{array} \right) = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ -r^* & 1 + r^* \end{array} \right) \left( \begin{array}{c} (k_t - k^*) / k^* \\ (q_t - q^*) / q^* \end{array} \right) + \left( \begin{array}{c} \frac{1}{2(1+r^*)} \\ 1 \end{array} \right) (r_t - r^*). \]

When \( r_t = r^* \), open loop dynamics (i.e. the system uses only the current state without a feedback) are given by (see appendix 1):

\[
\left( \begin{array}{c} \frac{k_{t+1} - k^*}{k^*} \\ \frac{q_{t+1} - q^*}{q^*} \end{array} \right) = \left( \begin{array}{c} k_0 - k^* \\ k^* \end{array} \right) \cdot \left( \frac{1}{2} \right)^{t+1} \left( \begin{array}{c} 1 \\ 2r^* \end{array} \right) \left( \begin{array}{c} 1 \\ 1 + 2r^* \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \left( \begin{array}{c} (k_t - k^*) / k^* \\ (q_t - q^*) / q^* \end{array} \right) + \left( \begin{array}{c} \frac{q_0 - q^*}{q^*} - \frac{2r^*}{1 + 2r^*} k_0 - k^* \\ \frac{2r^*}{1 + 2r^*} k^* \end{array} \right) \cdot (1 + r^*)^{t+1} \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \]

Loan to the current value of capital owned by borrowers is a linear function of the expected value of assets, so that the no-bubble assumption is equivalent to a no Ponzi game condition on debt:

\[
\frac{b_t}{q_t k_t} = \frac{1}{R_t} E_t \left( \frac{q_{t+1}}{q_t k_t} \right) \quad \text{and} \quad 0 \leq k_t \leq K < +\infty, \quad 1 \leq R_t \leq 2 \Rightarrow \lim_{s \to +\infty} E_t \left( \frac{q_{t+s}}{\prod_{\tau=0}^{s} R_\tau} \right) = 0 \iff \lim_{s \to +\infty} E_t \left( \frac{b_{t+s}}{\prod_{\tau=0}^{s} R_\tau} \right) = 0. \]
2.2. Type I equilibrium: Optimal Policy under Commitment to Financial Stability

In this section, we introduce an optimal policy feedback rule with commitment in the linear quadratic dynamic Stackelberg game with rational expectations (Ljungqvist and Sargent (2012), chapter 19). In a first step, the central bank determines the weights of an optimal rule minimizing a quadratic loss function in order to control the open loop dynamical system of the decentralized private sector with divergent rational expectations of asset prices:

\[
\max_{\{R_t\}} - \frac{1}{2} \sum_{t=0}^{\infty} \beta^t L = - \begin{pmatrix} \frac{k_0 - k^*}{k^*} \\ \frac{q_0 - q^*}{q^*} \end{pmatrix}^T P \begin{pmatrix} \frac{k_0 - k^*}{k^*} \\ \frac{q_0 - q^*}{q^*} \end{pmatrix} \quad \text{with:}
\]

\[
L = Q_{kk} \left( \frac{k_t - k^*}{k^*} \right)^2 + Q_{qq} \left( \frac{q_t - q^*}{q^*} \right)^2 + Q_{kq} \left( \frac{k_t - k^*}{k^*} \right) \left( \frac{q_t - q^*}{q^*} \right) + \rho (r_t - r^*)^2
\]

As the Central Bank is free to choose any discount factor, we may assume that she uses the same discount factor than the patient lenders, which is equal to the long-run interest rate on loans \( (\beta' = 1/(1 + r^*)) \). The cost of changing the policy interest rate (control variable) has to be strictly positive, although it can be relatively very small: \( R = \rho > 0 \) (with usual optimal control notation for the matrix \( R \) distinct from the interest rate factor \( R \)) is positive definite. The matrix \( Q \) weights the losses related to states variables \( (k_t, q_t) \): it is positive semidefinite. Setting \( Q = 0 \) is equivalent to a relatively infinitely larger cost of changing the policy interest rate \( R = \rho \) is a maximal inertia case stabilizing bubbles. The Central Bank may wish to decrease the covariance between asset prices and capital, in order to limit the wealth effect channel of financial instability to real instability (that is, limit the contagion effect of asset price fluctuations on output), so that \( Q_{kq} \neq 0 \), provided \( Q_{kk} > 0; Q_{qq} > 0 \) and \( Q_{kk}Q_{qq} - Q_{kq}^2 > 0 \). \( P \) is a symmetric matrix (when \( Q \) is symmetric) which provides the optimal value of the loss function (see appendix for its computation, as a solution of a discrete algebraic Ricatti equation).

In a first step suggested by Ljungqvist and Sargent (2012), it is assumed a policy feedback linear rule depending on borrowers capital and "as if" it depends on a non-predetermined variable (the current value of asset prices \( q_t \)) and, with feedback weights denoted \( \phi_k \) and \( \phi_q \). But, in theory, this interest rule depends only on predetermined variables (capital \( k_t \) and the Lagrange multiplier related to the non-predetermined variable \( \mu_{q,t} \)) with feedback weights denoted \( \phi_{k,\mu} \neq \phi_k \) and \( \phi_{\nu_q} \).
following the subsequent steps of the algorithm by Ljungqvist and Sargent (2012). Finally, the Lagrange multiplier $\mu_{q,t}$ can be substituted so that the optimal rule is as a history dependent rule depending on $k_t$, $k_{t-1}$, $r_{t-1}$, with weights $\phi_{k_t, t}$, $\phi_{k_{t-1}, t}$, $\phi_{r_{t-1}, r}$, using a similar insight by Ljungqvist and Sargent (2012) and Woodford’s (2003b) model with consumer prices.

$$
\begin{align*}
    r_t - r^* &= -\phi_k \left( \frac{k_t - k^*}{k^*} \right) - \phi_q \left( \frac{q_t - q^*}{q^*} \right) = -\phi_{k_t, \mu} \left( \frac{k_t - k^*}{k^*} \right) - \phi_{\mu, t} \mu_{q, t} \\
    r_t - r^* &= -\phi_{k_t, \mu} \left( \frac{k_t - k^*}{k^*} \right) - \phi_{k_{t-1}, \mu} \left( \frac{k_{t-1} - k^*}{k^*} \right) - \phi_{r_{t-1}, r} (r_t - r^*)
\end{align*}
$$

The policymaker policy choice is subject to (with usual matrix notations in bold letters):

$$
\begin{pmatrix}
    \frac{k_{t+1} - k^*}{q_{t+1} - q^*} \\
    \frac{1}{1 + r^*}
\end{pmatrix} =
\begin{pmatrix}
    \frac{1}{2(1+r^*)} \\
    1
\end{pmatrix} +
\begin{pmatrix}
    \phi_k & \phi_q \\
    \phi_{k_t, \mu} & \phi_{q_t, \mu}
\end{pmatrix}
\begin{pmatrix}
    \frac{k_t - k^*}{k^*} \\
    \frac{q_t - q^*}{q^*}
\end{pmatrix}
$$

The rank of the matrix $\sqrt{\beta} \mathbf{B} \mathbf{B}' \mathbf{A} \mathbf{B}$ which is the same than the rank of the matrix $(\mathbf{B} \mathbf{A} \mathbf{B})$ is equal to the number of variables in the system ($n = 2$):

$$
\begin{pmatrix}
    \sqrt{\beta} \mathbf{B} \mathbf{B}' \mathbf{A} \mathbf{B} \\
    \sqrt{\beta} \mathbf{B}' \mathbf{A} \mathbf{B}
\end{pmatrix} =
\begin{pmatrix}
    \frac{1}{2(1+r^*)} \\
    \frac{\beta'}{\sqrt{\beta}} \mathbf{B}' \left( 1 + r^* - \frac{r_t}{2(1+r^*)} \right)
\end{pmatrix}
$$

Hence, the open loop system is controllable. Firstly, this is equivalent to state that the Central Bank is always able to control asset price bubbles in this model: all closed loop eigenvalues are such that . Hence, as in Woodford (2003a), we may omit consideration of the transversality conditions (no bubbles on asset price), as we shall consider only bounded solutions, which necessarily satisfy the transversality conditions. Secondly, controllability imply that the condition on the Lagrange multiplier related to non pre-determined variables $\mu_{q, t=0} = 0$ is a valid initial condition for the optimal control problem (Bryson and Ho (1975), p.55-59; Xie (1997)):

$$
\mu_{q, t=0} = 0
$$
The stabilization is only a local result, valid for the linearized system following small permanent productivity shocks on $A$ or $a$ in the neighborhood of the long run equilibrium (say 10% deviation from their equilibrium for asset price and households loans). We assume the productivity shocks to be small enough, so that the optimal feedback rule does not lead to an interest rate which is below the zero lower bound or over the marginal productivity of borrowers ($0 < R_t < A$). Even with these small initial shocks close to equilibrium, asset price diverge to infinity (bubbles) or zero (fire sales) in the following period without negative feedback control.

At the initial date, the multiplier $\mu_{k,t}$ for the pre-determined variables jumps in order to satisfy the given constraint $k_0$ of their initial value. Conversely, the non-predetermined variables $q_t$ jumps at the initial date, in order to satisfy the constraint $\mu_{q,t=0} = 0$ at its initial condition. The optimal relation between multipliers and state variables is given by:

\[
\begin{pmatrix}
\mu_{k,t} \\
\mu_{q,t}
\end{pmatrix} = \begin{pmatrix}
P_{kk} & P_{kq} \\
P_{kq} & P_{qq}
\end{pmatrix} \begin{pmatrix}
k_t - k^* \\
q_t - q^*
\end{pmatrix} \Rightarrow

\frac{q_t - q^*}{q^*} = -P^{-1}_{qq} P_{qk} \left( \frac{k_t - k^*}{k^*} \right) + P^{-1}_{qq} \mu_{q,t} \Rightarrow

\frac{q_0 - q^*}{q^*} = -P^{-1}_{qq} P_{qk} \left( \frac{k_0 - k^*}{k^*} \right) \text{ and } \mu_{q,t=0} = 0
\]

When the open loop system is controllable, the latter condition determines the initial value of non pre-determined variables $q_0$ (Xie (1997)). Hence, there is no indeterminacy with optimal rules under commitment, by contrast with ad hoc rules which face Loisel’s (2009) dilemma between bubbles versus sunspots. Then, the optimal rule can be written as a function of the two pre-determined variables $k_t$ and $\mu_{q,t}$ at date $t$:

\[
r_t = \left( \begin{array}{cc}
\phi_k & \phi_q \\
-F
\end{array} \right) \begin{pmatrix}
I & 0 \\
-P^{-1}_{qq} P_{qk} & P_{qq}^{-1}
\end{pmatrix} \begin{pmatrix}
k_t - k^* \\
\mu_{q,t}
\end{pmatrix} = \begin{pmatrix}
f_{11} & f_{12} \\
& \mu_{q,t}
\end{pmatrix}
\]

The description of the Stackelberg plan is:
\[
\begin{pmatrix}
\frac{k_{t+1}-k^*}{k^*} \\
\mu_{q,t+1}
\end{pmatrix} =
\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
\frac{k_{t}-k^*}{k^*} \\
\mu_{q,t}
\end{pmatrix}
\text{with:}
\]
\[
\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix} =
\begin{pmatrix}
I & 0 \\
P_{kq} & P_{qq}
\end{pmatrix}
(A - BF)
\begin{pmatrix}
I \\
-P_{qq}^{-1}P_{kq}P_{qq}^{-1}
\end{pmatrix}
\]

As in Woodford (2003b): it is obvious that such an optimal plan will, in general, not be time consistent, in the sense discussed by Calvo (1978). For a policymaker that solves a corresponding problem starting at some date \(T = 0\) will choose processes for dates \(t \geq T\) that satisfy the above plan but starting with initial conditions such as \(\mu_{q,t} = 0\) again. Yet this last condition will, in general, not be satisfied by the above optimal plan chosen at date zero, for the evolution of the Lagrange multiplier. This is why discretionary optimization leads to a different equilibrium outcome than the one characterized here (see for example Miller and Salmon (1984a,b), Giordani and Söderlind (2004) and Blake and Kirsanova (2012)).

We can eliminate the implementation multipliers \(\mu_{q,t}\) in order to express the optimal rule as a function of \(k_t, k_{t-1},\) and \(r_{t-1}\): this is an history dependent representation of the decision rule:

\[
rt = \begin{pmatrix} f_{11} & f_{12} \end{pmatrix} \begin{pmatrix} \frac{k_{t}-k^*}{k^*} \\ \mu_{q,t} \end{pmatrix} \Rightarrow \mu_{q,t} = f_{12}^{-1} \left[ rt - f_{11} \begin{pmatrix} \frac{k_{t}-k^*}{k^*} \end{pmatrix} \right]
\]

\[
\mu_{q,t} = \begin{pmatrix} m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} k_{t-1}-k^* \\ k^* \end{pmatrix} + m_{22} \mu_{q,t-1} \text{ and } \mu_{q,t-1} = f_{12}^{-1} \left[ r_{t-1} - f_{11} \begin{pmatrix} \frac{k_{t-1}-k^*}{k^*} \end{pmatrix} \right]
\]

\[
r_{t} = f_{11} \begin{pmatrix} \frac{k_{t}-k^*}{k^*} \end{pmatrix} + f_{12} \begin{pmatrix} m_{21} & m_{22}f_{12}^{-1} \end{pmatrix} f_{11} \begin{pmatrix} \frac{k_{t-1}-k^*}{k^*} \end{pmatrix} + f_{12}m_{22}f_{12}^{-1} \cdot r_{t-1}
\]

As in Woodford (2003), the dependence of policy in period \(t\) on \(r_{t-1}\) can be a substitute for explicit dependence on the value of the multiplier \(\mu_{q,t-1}\) (a variable which has no meaning outside the context of the planning problem considered here).

In table 1, using numerical values: \(r^* = 1.03\) and the LQR Scilab code, we present, for different weights \(R\) and \(Q\) in the loss function, the optimal weights \(F\) of the augmented Taylor rule and the eigenvalues of the closed loop system:

Table 1: Optimal feedback rules and eigenvalues of optimal policy under commitment
<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$Q_{kk}$</th>
<th>$Q_{qq}$</th>
<th>$Q_{kq}$</th>
<th>$P_{kk}$</th>
<th>$P_{QQ}$</th>
<th>$P_{kq}$</th>
<th>$\phi_k$</th>
<th>$\phi_q$</th>
<th>$\lambda_k$</th>
<th>$\lambda_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>1.03</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>0</td>
<td>0</td>
<td>0.507</td>
<td>1.045</td>
</tr>
<tr>
<td>&gt;0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0003048</td>
<td>0.0951066</td>
<td>-0.0053838</td>
<td>-0.000545</td>
<td>0.0898</td>
<td>0.507</td>
<td>0.955</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.0010875</td>
<td>1.6782322</td>
<td>-0.0247239</td>
<td>-0.023648</td>
<td>0.6536</td>
<td>0.532</td>
<td>0.368</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1.2783707</td>
<td>0.1841146</td>
<td>-0.0767203</td>
<td>0.193604</td>
<td>0.1094</td>
<td>0.389</td>
<td>0.956</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1.3543311</td>
<td>1.9195623</td>
<td>-0.3014750</td>
<td>0.042778</td>
<td>0.6261</td>
<td>0.604</td>
<td>0.292</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1.3543311</td>
<td>1.9195623</td>
<td>-0.3014750</td>
<td>0.042778</td>
<td>0.6261</td>
<td>0.604</td>
<td>0.292</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.0844313</td>
<td>1.4231479</td>
<td>1.1228422</td>
<td>0.204715</td>
<td>0.5420</td>
<td>0.194</td>
<td>0.707</td>
</tr>
</tbody>
</table>

On table 1, the first row provides the numerical values of the eigenvalues of the matrix \( A \) open loop model of the previous section: 0.5 for convergent capital owned by borrowers and 1.03 for divergent asset prices. The second row provides the eigenvalues of the open loop model for the matrix \( 1/\sqrt{\beta}A \) to take into account discounting when solving discrete algebraic Riccati equation which are slightly larger: 0.5074 and 1.0453.

Central bank stabilizes asset prices only (maximal interest rate smoothing or minimal variance of its interest rate): In table 1, the third row presents the minimal effort of control of the economy by the Central Bank using the LQR. The weights related to state variables are all equal to zero \( Q = 0 \) and the weight related to the control variable is equal to one \( \rho = 1 \). This case is a limit case of a very small values in the weights \( Q \) with respect to a very large values of \( \rho \) which is the cost of changing the interest rate with respect to its long run value (it is distinct but related to another assumption found in the literature which emphasizes costs of changing the interest rate by the Central Bank). The loss function is then a function of the square of the deviations of state variables weighted by the Taylor rules parameters:

\[
\max_{\{\phi_k, \phi_q\}} -\frac{\rho}{2} \sum_{t=0}^{+\infty} \beta^t (r_t - r^*)^2 = -\frac{\rho}{2} \sum_{t=0}^{+\infty} \beta^t (\phi_k (k_{t+1} - k^*) - \phi_q (q_{t+1} - q^*))^2
\]

If \( Q = 0 \), any other strictly positive value for \( \rho > 0 \) leads to identical results for the optimal Taylor rule. This minimal effort loss function leads to the Taylor rule with the smallest weights that stabilizes asset prices.
\[ r_t = 0.00055 \left( \frac{k_t - k^*}{k^*} \right) - 0.0898 \left( \frac{q_t - q^*}{q^*} \right) \]

\[ r_t = -0.0045 \left( \frac{k_t - k^*}{k^*} \right) - 0.944 \cdot \mu_{q,t} \]

\[ r_t = -0.0045 \left( \frac{k_t - k^*}{k^*} \right) + 0.0043 \left( \frac{k_{t-1} - k^*}{k^*} \right) + 0.956 \cdot r_{t-1} \]

In this case, the convergent eigenvalues do not change after stabilization \((\lambda_k = 0.5074)\), but the other closed loop eigenvalues of \(A - BF\) are the mirrored (with respect to the unit circle) unstable open loop eigenvalues of \(A\) \((\lambda_q = 0.9547 = 1/1.0453)\). This eigenvalue is very close to the auto-regressive value of the history dependent rule \((\phi_{r_{t-1}} = 0.956)\). In table 1, this case corresponds to the largest closed loop stable eigenvalue for asset prices \(\lambda_q\) and the lowest coefficients in absolute values for asset prices in the Taylor rule \((\phi_q = -0.089)\). This means that a 10% increase of asset price from its fundamental value along (within the range of the local approximation, along with a 10% increase of capital) is related to an increase of interest rate at the value of \(r_t = 3.9\%\) to be compared to its long run value \(r^* = 3\%\).

The interpretation of a slightly procyclical coefficient for capital in the Taylor rule is \(\phi_k = +0.00055\) is erroneous, because in fact the convergence rate of capital towards equilibrium (measured by its eigenvalue \(\lambda_k\)) is unchanged, so that monetary policy is neither pro-cyclical nor counter-cyclical with respect to borrowers capital. Appendix B compares the explicit solution of the closed loop dynamical system. The eigenvector does not change for the stable open loop eigenvalue related to capital (hence, in a phase diagram, the line \(q_t = q_{t+1}\) is changed. The eigenvector for the unstable open eigenvalue changes very marginally (hence, in a phase diagram, the line \(k_t = k_{t+1}\) shift from an open loop vertical line, to a nearly vertical line with a very high negative slope). The phase diagram around the equilibrium is quasi-identical to the one of Miller and Stiglitz (2010), except that in our case it is a sink where all paths are converging to the equilibrium.

Finally, we have the following description of the Stackelberg plan with capital and the Lagrange multiplier for asset prices, the optimal path for asset price and its initial condition:
\[
\begin{pmatrix}
\frac{k_{t+1} - k^*}{k^*} \\
\mu^t_{q,t+1}
\end{pmatrix} =
\begin{pmatrix}
0.50744 & -0.46518 \\
1.9413 \times 10^{-8} & 0.95666
\end{pmatrix}
\begin{pmatrix}
\frac{k_t - k^*}{k^*} \\
\mu^t_{q,t}
\end{pmatrix}
\]

with:
\[
\frac{q_t - q^*}{q^*} = 0.057 \cdot \left( \frac{k_t - k^*}{k^*} \right) + 10.51 \cdot \mu_{q,t}
\]
\[
\frac{q_0 - q^*}{q^*} = -P_{qq}^{-1}P_{qk} \left( \frac{k_t - k^*}{k^*} \right) = 0.057 \cdot \left( \frac{k_0 - k^*}{k^*} \right)
\]

because \( \mu_{q,0} = 0 \)

The fourth row adds a weight only for the benefits of stabilizing asset prices \( Q_{kk} = 0, Q_{kq} = Q_{qq} = 1 \). In this case, capital is slightly taking more time to converge \( (\lambda_k = 0.532 > 0.507) \) with a rule parameter \( \phi_k = -0.02 \), while the rule parameter \( \phi_q = 0.65 \), with a sharp increase of convergence of asset prices \( (\lambda_q = 0.368 < 0.956 < 1.045) \).

**Central Bank stabilizes output and asset prices:** The fifth row adds a weight only for the benefits of stabilizing capital \( Q_{kk} = 1, Q_{kq} = Q_{qq} = 0 \). In this case, the coefficient on capital in the rule is now \( \phi_k = 0.19 \) with an increase of convergence \( (\lambda_k = 0.389 < 0.507) \). The coefficient of asset price in the rule is \( \phi_q = 0.109 \) with the same speed of convergence than in the maximal inertia case \( (\lambda_q = 0.956) \).

The fifth row adds an identical weight for stabilizing capital and asset prices \( Q_{kk} = 1, Q_{kq} = 0, Q_{qq} = 1 \). In this case, the coefficient on capital in the rule is now \( \phi_k = 0.043 \) with a decrease of convergence \( (\lambda_k = 0.604 > 0.507) \). The coefficient of asset price in the rule is \( \phi_q = 0.626 \) with a sharp increase of convergence \( (\lambda_q = 0.292 < 0.956) \).

**Central Bank decreases the asset price channel to output (wealth effects):** The sixth row adds a weight only on the covariance between capital and asset prices \( Q_{kk} = 0, Q_{kq} = 1, Q_{qq} = 0 \). The results are identical to the fifth row with an identical weight for stabilizing capital and asset prices \( Q_{kk} = 1, Q_{kq} = 0, Q_{qq} = 1 \). Finally, the seventh row adds an equal weight to elements of the variance covariance matrix between capital and asset prices \( Q_{kk} = 1, Q_{kq} = 1, Q_{qq} = 1 \). In this case, the coefficient on capital in the rule is now \( \phi_k = 0.204 \) with an increase of convergence \( (\lambda_k = 0.194 < 0.507) \). The coefficient of asset price in the rule is \( \phi_q = 0.542 \) with a sharp increase of convergence \( (\lambda_q = 0.707 < 0.956) \).

Depending on weights in the loss function, a single instrument is able to stabilize two policy state variables: capital and asset prices (cf. Blanchard [2012] on the current debate on macroprudential instruments and monetary policy). Tin-
bergen’s rule is not necessarily valid for the linear quadratic regulator: one control variable can stabilize more than one state variable.

2.3. Type II Equilibrium

If the private sector assumes no bubbles on asset prices (Kiyotaki and Moore (1997) assumption 3 (no bubbles on asset prices)) before the policymaker decision, this implies that, if the initial condition \( k_0 \) and \( q_0 \) is such that \( \frac{q_0 - q^*}{q^*} \neq \frac{2r^* - k_0 - k^*}{k^*} \), the asset price variable \( q_0 \) jumps instantaneously at the value \( q'_0 \) such that \( \frac{2r^* - k_0 - k^*}{k^*} \). Hence, the dynamics of assets prices are restricted to be convergent on the stable manifold of the saddlepoint equilibrium determined by \( \frac{q_{t+1} - q^*}{q^*} = \frac{2r^* - k_{t+1} - k^*}{k^*} = \frac{r^* - k_t - k^*}{q^*} \). The rank of matrix of the system is now one instead of two:

\[
\begin{pmatrix}
\frac{k_{t+1} - k^*}{k^*} \\
\frac{q_{t+1} - q^*}{q^*}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 0 \\
\frac{r^*}{1 + 2r^*} & 0
\end{pmatrix} \begin{pmatrix}
\frac{k_t - k^*}{k^*} \\
\frac{q_t - q^*}{q^*}
\end{pmatrix}
\]

Taking into account the discount rate in the loss function, the closed loop system is controllable when the eigenvalues of the closed loop matrix \( \sqrt{\beta} (A - BF) \) are all with modulus lower than 1 (or the \( A - BF \) has eigenvalues all below \( 1/\sqrt{\beta} = \sqrt{R^*} \). The system is not completely state controllable because Kalman condition that the following controllability matrix has a rank equal to 2 is not fulfilled. Indeed, a policy feedback rule depending on asset prices and capital boils down to a feedback rule depending only on capital:

\[
\begin{align*}
\left( r_t - r^* \right) &= -\phi_k \left( \frac{k_t - k^*}{k^*} \right) - \phi_q \left( \frac{q_t - q^*}{q^*} \right) \\
&= -\left( \phi_k + \frac{2r^*}{1 + 2r^*} \phi_q \right) \left( \frac{k_t - k^*}{k^*} \right).
\end{align*}
\]

Adding non pre-determined variables in the policy feedback rule is qualitatively useless. The policymaker optimal program is then reduced to the pre-determined variable or to the non pre-determined variable:
3. General Case

3.1. Type I Equilibrium: Optimal Policy under Central Bank Commitment to Financial Stability

This equilibrium is detailed in Ljungqvist and Sargent (2012). The policy maker as a Stackelberg leader commits to a sequence of decision rule at time 0. Her Stackelberg problem is to minimize her loss function by finding a sequence of decision rules \( r_t \).

\[
\max_{\{r_t\}, k_{t+1}, q_{t+1}} - \frac{1}{2} \sum_{t=0}^{+\infty} \beta^t \left( Q_{kk} + \rho \phi_k^2 \left( \frac{k_t - k^*}{k^*} \right)^2 \right) \\
= - \left( \begin{array}{c} k_0 - k^* \\ q_0 - q^* \end{array} \right)^T P \left( \begin{array}{c} k_0 - k^* \\ q_0 - q^* \end{array} \right)
\]
subject to a closed loop dynamics including the feedback rule:

\[
\begin{pmatrix}
  k_{t+1} \\
  q_{t+1}
\end{pmatrix} = 
\begin{pmatrix}
  A_{nn} & A_{nm} \\
  A_{mn} & A_{mm}
\end{pmatrix} + 
\begin{pmatrix}
  B_{n1} \\
  B_{m1}
\end{pmatrix}
\begin{pmatrix}
  \phi_{1n} & \phi_{1m} \phi_{1m} \\
  -F
\end{pmatrix}
\begin{pmatrix}
  k_t \\
  q_t
\end{pmatrix} + \gamma z_t
\] (7)

where \( k_t \) is an \((n \times 1)\) vector of variables predetermined at \( t \) with initial conditions \( k_0 \) given (shocks can straightforwardly be included into this vector); \( q \) is an \((m \times 1)\) vector of variables non-predetermined at \( t \); \( Z \) is an \((k \times 1)\) vector of exogenous variables; \( r_t \) is a \( p \times (n + m)\) vector of policy instruments of the policymaker with a linear policy feedback rule \(-F\); \( A \) is \((n + m) \times (n + m)\) matrix, \( \gamma \) is a \((n + m) \times k\) matrix, \( B \) is a \((n + m) \times p\), \( F \) is a \( p \times (n + m)\) matrix, \( \gamma q_t \) is the agents expectations of \( q_{t+1} \) defined as follows:

\[
\gamma q_{t+1} = E_t(q_{t+1} | \Omega_t)
\] (8)

\( \Omega_t \) is the information set at date \( t \) (it includes past and current values of all endogenous variables and may include future values of exogenous variables). According to Blanchard and Kahn (1980), a predetermined variable is a function only of variables known at date \( t \) so that \( k_{t+1} = \gamma k_{t+1} \) whatever the realization of the variables in \( \Omega_{t+1} \). A non-predicted variable can be a function of any variable in \( \Omega_{t+1} \), so that we can conclude that \( q_{t+1} = \gamma q_{t+1} \) only if the realization of all variables in \( \Omega_{t+1} \) are equal to their expectations conditional on \( \Omega_t \).

Boundary conditions for the policymaker first order conditions are the hypothesis of no bubbles for \( k, q \) and \( z \), and initial conditions for pre-determined variables \( k_0 \) but \( q_0 \) is to be chosen according to the constraint on its shadow price \( \mu_{q,t=0} \) at the initial date (see Bryson (1975, p.55-56) and Xie (1997)):

\[
k_0 \text{ given and } \lim_{s \to +\infty} E_t \left( \frac{z_{t+s}}{\prod_{\tau=0}^{s} (1 + r_\tau)} \right) = 0
\] (9)

An open loop system is controllable as soon as the Kalman condition on the rank of the following matrix is satisfied. It implies that the Lagrange multiplier related to non-predicted variables is zero.

\[
\text{rank } (B \ AB \ A^2B \ ... \ A^{n-1}B) = n \Rightarrow \mu_{q,t=0} = 0
\]

Because the open loop system is controllable, we may omit consideration of the transversality conditions (no bubbles on asset price), as we shall consider only
bounded solutions, which necessarily satisfy the transversality conditions; with all eigenvalues inside the unit circle (Woodford (2003)). The following transversality conditions are no longer necessary because non predetermined variables are bounded in the closed loop optimal control:

\[
\lim_{s \to +\infty} E_t \left( \frac{q_{t+s}}{\prod_{\tau=0}^{s} (1 + r_{\tau})} \right) = 0
\]

Ljungqvist and Sargent (2012, chapter 19) describe a four step algorithm for solving the Stackelberg problem. The solution exhibit a history dependent optimal rule, optimal paths are determinate for non pre-determined variables, and the solutions are locally bounded with local stability.

The minimal volatility of the policymakers interest rate is given by the case \((R > 0, Q = 0)\). It is such that stable eigenvalues of the open loop system are the same than in the closed loop system, unstable eigenvalues of the open loop system are mirrored as having their modulus \(\lambda_{CL} = 1/|\lambda_{OL}| < 1\).

### 3.2. Type II equilibrium: No-bubble assumption and identification problems.

Several DSGE assumes simultaneously (1) ad hoc augmented Taylor rules which are not derived from an optimal program by the Central Bank and (2) no asset price bubbles and no Ponzi game condition for credit with Blanchard and Kahn (1980) solutions. In this case, the parameters in the augmented Taylor rule may not be identified. To understand this issue, we present how may work an optimal rule decision when the private sector is able to rule out bubbles.

Before setting her policy feedback rule, the policymaker assumes that the private sector behaves assuming no bubbles (or no Ponzi game) for non predetermined variables. This describes "jumps" of non predetermined variables on the stable manifold originated by pre-determined variables. Then, the orthogonalized expectational non pre-determined variables (denoted \(q_t\)) with explosive eigenvalues are linear combinations of orthogonalized pre-determined variables with convergent eigenvalues (Blanchard and Kahn (1980) equation A6, p.1310, McCandless (2008), section 6.8 and Cochrane (2011), appendix B):

\[
q_{t+1} = -N_{mn} k_{t+1} - L [\varepsilon_{t+1}] \text{ and } E_t [\varepsilon_{t+1}] = 0
\]

\[
\Rightarrow t_0 q_{t+1} = -N_{mn} k_{t+1}
\]
The formal derivations of the matrix $N$ and $L$ in the general case including stochastic shocks and when the generalized Schur method is necessary are presented in McCandless (2008), section 6.8. If the Central Bank defines a rule on both pre-determined and non pre-determined variables, it is observationally equivalent to a rule which depends only on pre-determined variables with weights $\phi'_{1n}$ with all weights of non pre-determined variables equal to zero ($\phi'_{1m} = 0$).

$$r_t - r^* = -\phi_{1n} \left( \frac{k_{t+1} - k^*}{k^*} \right) - \phi_{1m} \left( \frac{q_{t+1} - q^*}{q^*} \right)$$

$$= -\left( \phi_{1n} - \phi_{1m}N_{nn} \right) \left( \frac{k_{t+1} - k^*}{k^*} \right).$$

$$\phi'_{1n} = \phi_{1n} - \phi_{1m}N_{nn}$$

The policymaker controls pre-determined variables and then, she will control non pre-determined expectational variables dynamics according to $r_{t+1} = -N_{mn}k_{t+1}$:

$$\max_{\{R_t\}} \left[ \frac{1}{2} \sum_{t=0}^{+\infty} \beta^t \left[ Q_{nn} \left( \frac{k_t - k^*}{k^*} \right)^2 + \rho (r_t - r^*)^2 \right] \right]$$

subject to the stable part of the closed loop system:

$$k_{t+1} = [A_{nn} - A_{nm}N_{mn} + B_{n1}\phi'_{1n}] k_t$$

which corresponds to the top half of partitioned matrices of the full (non-controllable) closed loop system:

$$\begin{pmatrix} k_{t+1} \\ \tau q_{t+1} \end{pmatrix} = \begin{pmatrix} I_{nn} \\ -N_{nn} \end{pmatrix} \begin{pmatrix} k_{t+1} \\ k_{t+1} \end{pmatrix}$$

$$= \left( \begin{pmatrix} A_{nn} & A_{nm} \\ A_{mn} & A_{mm} \end{pmatrix} \begin{pmatrix} I_{nn} \\ -N_{nn} \end{pmatrix} + \begin{pmatrix} B_{n1} \\ B_{m1} \end{pmatrix} \begin{pmatrix} \phi_{1n} & \phi_{1m} \end{pmatrix} \begin{pmatrix} I_{nn} \\ -N_{nn} \end{pmatrix} \right) \begin{pmatrix} k_{t+1} \\ k_t \end{pmatrix}$$ (12)

$$= \begin{pmatrix} k_{t+1} \\ \tau q_{t+1} \end{pmatrix}$$ (13)

Hence, including non pre-determined variables in the policy rule implies that their parameters are not identified (see Cochrane (2011), appendix B, pages 10-11). Because pre-determined variables are backward looking, all policies are time consistent in this simple model.
This model is solved using the backward looking linear quadratic regulator. It decreases the modulus of eigenvalues of pre-determined variables when $Q_{nn} > 0$. Type II equilibrium faces three drawbacks:

(1) As mentioned above, one is unable to identify parameters of non pre-determined variables (such as asset prices) if ever some of those variables are included in the optimal rule.

(2) Determinacy is not granted. For example, Cochrane [2011] insists that rational expectations bubbles for consumer prices are possible non local multiple equilibria and that the no-bubble condition did not come from any economics of the model. He concludes that the new Keynesian model are not able to determine price, once the no-bubble condition is not assumed. Then, he investigates that various arguments or formal theories proposed so far in the case of consumer price bubbles. For example, the learning theory is not always able to justify determinacy: bubbles can be learned and not the stable path of saddlepoint equilibria in Cochrane [2009]. The same critique is likely to hold for the no-bubble condition on asset price, but it would require a lengthy specific paper to investigate the overall literature.

(3) Local instability. If there is an infinitesimal deviation from the stable manifold due to a private sector or policymaker errors, asset price on unstable path will quickly deviate outside the boundaries where the linear approximation of the system is valid (say over 10% of the fundamental value). If an economy were to find itself on a divergent path, the information should be extremely precise to jump back again on the stable manifold (Burmeister [1980]).

4. Conclusion

Central Banks with optimal feedback rules with commitment to financial stability can stabilize divergent asset prices or divergent private debt, and identify parameters in a history dependent feedback rule. Many extensions are feasible.

References


5. Appendix

Appendix 1: Linearization of the open loop model.

We linearize the dynamical system around this equilibrium.

\[
\frac{\partial f_1}{\partial k_t}(k^*, q^*, R^*) = \sqrt{\frac{RA}{a}} \frac{1}{2 \sqrt{k_t^*}} = \frac{1}{2} \\
\frac{\partial f_1}{\partial R_t}(k^*, q^*, R^*) = \sqrt{\frac{Ak^*}{a}} \frac{1}{2 \sqrt{R^*}} = \sqrt{\frac{AR^*A}{a^2}} \frac{1}{2 \sqrt{R^*}} = \frac{A}{2a} = \frac{k^*}{2R^*} \\
\frac{\partial f_2}{\partial R_t}(k^*, q^*, R^*) = q^*
\]

Log linearisation for capital and asset price (and not for interest rate):

\[
k_{t+1} - k^* = \frac{1}{2} (k_{t} - k^*) + \frac{k^*}{2R^*} (r_t - r^*) \\
k_{t+1} - k^* = \frac{1}{2} \left( \frac{k_{t} - k^*}{k^*} \right) + \frac{1}{2R^*} (r_t - r^*) \\
q_{t+1} - q^* = -a (k_{t} - k^*) + (1 + r^*) (q_{t} - q^*) + q^* (r_t - r^*) \\
q_{t+1} - q^* = -\frac{ak^*}{q^*} \left( \frac{k_{t} - k^*}{k^*} \right) + (1 + r^*) \left( \frac{q_{t} - q^*}{q^*} \right) + (r_t - r^*) \\
\frac{ak^*}{q^*} = \frac{aR^*Ar^*}{aR^*} = r^*
\]

This leads to the system:

\[
\begin{pmatrix}
(k_{t+1} - k^*) / k^* \\
(q_{t+1} - q^*) / q^*
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{2} & 0 \\
-r^* & 1 + r^*
\end{pmatrix}
\begin{pmatrix}
(k_{t} - k^*) / k^* \\
(q_{t} - q^*) / q^*
\end{pmatrix}
+ 
\begin{pmatrix}
\frac{1}{2(1+r^*)} \\
1
\end{pmatrix}
(r_t - r^*).
\]

The Jordan form of the open loop matrix, with its eigenvectors matrix \( M \), is equal to:
\[ A = \begin{pmatrix} \frac{1}{2} & 0 \\ -r^* & 1 + r^* \end{pmatrix} = \text{MDM}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{2r^*}{1+2r^*} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & r^* + 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{1+2r^*} & 0 \\ 1 & 1 \end{pmatrix} \]

The solutions of the open loop dynamical system as a function of initial conditions \( k_0 \) and \( q_0 \):

\[
\begin{pmatrix} \frac{k_{t+1}-k^*}{k^*} \\ \frac{q_{t+1}-q^*}{q^*} \end{pmatrix} = \text{MDM}^{-1} \begin{pmatrix} \frac{k_t-k^*}{k^*} \\ \frac{q_t-q^*}{q^*} \end{pmatrix} = \text{MD}^{t+1} \text{M}^{-1} \begin{pmatrix} \frac{k_0-k^*}{k^*} \\ \frac{q_0-q^*}{q^*} \end{pmatrix}
\]

\[
\text{MD}^{t+1} \begin{pmatrix} \frac{k_0-k^*}{k^*} \\ \frac{q_0-q^*}{q^*} \end{pmatrix} = \text{MD}^{t+1} \begin{pmatrix} \frac{k_0-k^*}{k^*} \\ \frac{q_0-q^*}{q^*} - \frac{2r^*}{1+2r^*} \right) \begin{pmatrix} \frac{k_0-k^*}{k^*} \\ \frac{q_0-q^*}{q^*} - \frac{2r^*}{1+2r^*} \end{pmatrix} \end{pmatrix} \cdot (1 + r)^{t+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

**Appendix 2: Solution of Type I equilibrium**

We follow exactly Ljundqvist and Sargent’s (2012, Chapter 19) four steps. The Lagrangean of the Stackelberg problem is:

\[ -\frac{1}{2} E_t \sum_{t=0}^{\infty} \beta^t \left\{ x_t^T Q x_t + r_t^T R r_t - 2\beta^t \mu_{t+1}^T (x_{t+1} - A x_t - B r_t) \right\} \]  

(14)

where \( 2\beta^t \mu_{t+1} \) is the Lagrange multiplier associated with the linear constraint. First order conditions with respect to \( r_t \) and \( x_t \), respectively, are:

\[
\begin{align*}
0 &= \text{RR}_t + \beta^t \text{B}^T E_t (\mu_{t+1}) \\
\mu_t &= \text{Q} x_t + \beta^t \text{A}^T E_t (\mu_{t+1})
\end{align*}
\]

(15) \hspace{1cm} (16)

The stabilization matrix \( P \) is solution of the discounted discrete algebraic Riccati equation:

\[
P = Q + \beta^t A^T P A - \beta^{2t} A^T P B \left( R + \beta^t B^T P B \right)^{-1} B^T P A.
\]

Knowing \( P \), one finds the optimal rule feedback matrix \( F \):

\[
F = \beta^t \left( R + B^T P B \right)^{-1} B^T P A
\]

(23)
Knowing $F$, one finds the closed loop matrix $A - BF$ and a final algorithm can compute its stable eigenvalues denoted $\lambda_k$ and $\lambda_q$. The stabilization matrix $P = P > 0$ is solution of the discounted discrete algebraic Riccati equation (Ljungqvist L. and Sargent T.J. (2012)) with $\beta \geq 1$, $B/\sqrt{\beta} = 1/\sqrt{\beta}$, $A/\sqrt{\beta} = (1 + r^*) / \sqrt{\beta}$.

**Appendix 3: Closed Loop Numerical computation**

We set as a benchmark case: $r^* = 0.03$. The matrix $A$ and $B$ multiplied by $1/\sqrt{\beta} = \sqrt{1 + r^*}$ are equal to. Jordan form of the closed loop matrix, for the Taylor rule with smallest weights ($Q = 0$):

$$
\begin{pmatrix}
0.50744 & 0 \\
-0.030447 & 1.0453
\end{pmatrix} + \begin{pmatrix}
0.49266 & 0.0050836 \\
1.0149 & -0.0898032
\end{pmatrix}
= \begin{pmatrix}
0.50994 & -0.044242 \\
-0.025288 & 0.95416
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
0.056608 & -10.097
\end{pmatrix} \begin{pmatrix}
0.50744 & 0 \\
0 & 0.95666
\end{pmatrix}
\begin{pmatrix}
0.99443 & 0.098484 \\
0.005575 & -0.098484
\end{pmatrix}
\begin{pmatrix}
k_0 & q_0 \\
k^* & q^*
\end{pmatrix}
\begin{pmatrix}
k_{t+1} - k^* \\
q_{t+1} - q^*
\end{pmatrix}
$$

The solutions of the closed loop dynamical system as a function of initial conditions $k_0$ and $q_0$:

$$
\begin{pmatrix}
\frac{k_{t+1} - k^*}{k^*} \\
\frac{q_{t+1} - q^*}{q^*}
\end{pmatrix} = M_c D_c^{t+1} \begin{pmatrix}
0.99443 & 0.098484 \\
0.005575 & -0.098484
\end{pmatrix} \begin{pmatrix}
\frac{k_0 - k^*}{k^*} \\
\frac{q_0 - q^*}{q^*}
\end{pmatrix}
\begin{pmatrix}
\frac{k_0 - k^*}{k^*} \\
\frac{q_0 - q^*}{q^*}
\end{pmatrix}
\begin{pmatrix}
\frac{k_{t+1} - k^*}{k^*} \\
\frac{q_{t+1} - q^*}{q^*}
\end{pmatrix}
$$

To be compared to the open loop dynamical system with numerical values:

$$
\frac{2r^*}{1 + 2r^*} = \frac{0.06}{1 + 0.06} = 0.056608
$$

That is:

$$
\begin{pmatrix}
\frac{k_{t+1} - k^*}{k^*} \\
\frac{q_{t+1} - q^*}{q^*}
\end{pmatrix} = \left(\frac{k_0 - k^*}{k^*}\right) \cdot (0.5)^{t+1} \begin{pmatrix}
1 \\
0.056608
\end{pmatrix} + \left[-0.056608 \left(\frac{k_0 - k^*}{k^*}\right) + \left(\frac{q_0 - q^*}{q^*}\right)\right] \cdot (1.03)^{t+1}
$$
The eigenvector is the same for the stable open loop variables \((k)\) and it is nearly the same for the unstable open loop asset price variable turned to be stabilized in the closed loop system \((q)\): it is related to a vertical line in the phase diagram of the open loop versus a quasi vertical line with negative slope in the plane \((k, q)\).

At the initial date, the multiplier \(\mu_{k,t}\) for the pre-determined variables jumps in order to satisfy the given constraint \(k_0\) of their initial value. Conversely, the non-predetermined variables \(q_t\) jumps at the initial date, in order to satisfy the constraint \(\mu_{q,t=0} = 0\) at its initial condition.

\[
\begin{pmatrix}
\mu_{k,t} \\
\mu_{q,t}
\end{pmatrix} = \begin{pmatrix} P_{kk} & P_{kq} \\ P_{kq} & P_{qq} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ q_t - q^* \end{pmatrix} \Rightarrow
q_t - q^* = -P_{qq}^{-1} P_{kq} \left(\frac{k_t - k^*}{k^*}\right) + P_{qq}^{-1} \mu_{q,t}
\]

\[
\begin{align*}
\frac{q_t - q^*}{q^*} &= 0.057 \cdot \left(\frac{k_t - k^*}{k^*}\right) + 10.51 \cdot \mu_{q,t} \\
\frac{q_0 - q^*}{q^*} &= -P_{qq}^{-1} P_{kq} \left(\frac{k_t - k^*}{k^*}\right) = 0.057 \cdot \left(\frac{k_0 - k^*}{k^*}\right) \text{ because } \mu_{q,0} = 0
\end{align*}
\]

with:

\[
\begin{pmatrix} k_{t+1} - k^* \\ \mu_{q,t+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ P_{kq} & P_{qq} \end{pmatrix} (A - BF) \begin{pmatrix} I & 0 \\ -P_{qq}^{-1} P_{kq} & P_{qq}^{-1} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ \mu_{q,t} \end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
-0.0053838 & 0.0951066
\end{pmatrix} \begin{pmatrix} 0.50994 & -0.044242 \\
-0.025288 & 0.95416 \end{pmatrix} \begin{pmatrix}
-0.0053838 & 0 \\
0.0951066 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0.50744 & -0.46518 \\
1.9413 \times 10^{-8} & 0.95666
\end{pmatrix}
\]

Then, the optimal rule can be written as a function of the two pre-determined variables \(k_t\) and \(\mu_{q,t}\) at date \(t\):
\[ r_t = \begin{pmatrix} \phi_k & \phi_q \end{pmatrix} \begin{pmatrix} I & 0 \\ -P_{qq}^{-1} P_{pk} & P_{qq}^{-1} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ k^* \mu_{q,t} \end{pmatrix} \]

\[ r_t = \begin{pmatrix} 0.000545 \\ -0.0898 \end{pmatrix} \begin{pmatrix} -0.005383 & 0 \\ 0.0951066 & 1 \end{pmatrix} \begin{pmatrix} k_t - k^* \\ k^* \mu_{q,t} \end{pmatrix} \]

\[ r_t = \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ k^* \mu_{q,t} \end{pmatrix} = \begin{pmatrix} -4.5384 \times 10^{-3} \\ -0.9442 \end{pmatrix} \begin{pmatrix} k_t - k^* \\ k^* \mu_{q,t} \end{pmatrix} \]

Then, we can eliminate the implementation multipliers \( \mu_{q,t} \) in order to express the optimal rule as a function of \( k_t, k_{t-1}, \) and \( r_{t-1} \): this is an history dependent representation of the decision rule:

\[ r_t = \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix} \begin{pmatrix} k_t - k^* \\ k^* \mu_{q,t} \end{pmatrix} \Rightarrow \mu_{q,t} = f_{12}^{-1} \left[ r_t - f_{11} \left( \frac{k_t - k^*}{k^*} \right) \right] \]

\[ \mu_{q,t} = m_{21} \frac{k_{t-1} - k^*}{k^*} + m_{22} \mu_{q,t-1} \text{ and } \mu_{q,t-1} = f_{12}^{-1} \left[ r_{t-1} - f_{11} \left( \frac{k_{t-1} - k^*}{k^*} \right) \right] \]

\[ r_t = f_{11} \left( \frac{k_t - k^*}{k^*} \right) + f_{12} \left( m_{21} - m_{22} f_{12}^{-1} f_{11} \right) \left( \frac{k_{t-1} - k^*}{k^*} \right) + f_{12} m_{22} f_{12}^{-1} r_{t-1} \]

\[ r_t = -0.0045 \left( \frac{k_t - k^*}{k^*} \right) + 0.0043 \left( \frac{k_{t-1} - k^*}{k^*} \right) + 0.956 \cdot r_{t-1} \]

with:

\[ f_{12} \left( m_{21} - m_{22} f_{12}^{-1} f_{11} \right) = -0.9442 \cdot \left( 1.9413 \times 10^{-8} - 0.95666 \frac{-4.5384 \times 10^{-3}}{-0.9442} \right) \]

Replicating the numerical results is timeless following those steps: download the open source software Scilab, copy the following code into a text editor, change numerical values for matrix \( 1/\sqrt{\beta} \mathbf{A}, 1/\sqrt{\beta} \mathbf{B}, \mathbf{Q}, \) and \( \mathbf{R}, \) copy in the Scilab console window, obtain results for the solution of the discrete algebraic Ricatti equation \( \mathbf{P}, \) the optimal feedback rule matrix \( -\mathbf{F} \) and the eigenvalues of the closed loop system \( \mathbf{A} + \mathbf{B} (-\mathbf{F}). \)

\[
\begin{align*}
\mathbf{A}= & [0.50744, 0; -0.030447, 1.0453]; \\
\mathbf{B}= & [0.49266, 1.0149]; \\
\mathbf{Q}= & [1.0.1; 0.1, 0.0003]; \\
\mathbf{R}= & 1; \\
\text{Big}= & \text{sysdiag} (\mathbf{Q}, \mathbf{R}); \\
\end{align*}
\]
[w,wp]=fullrf(Big); C1=wp(:,1:2); D12=wp(:,3:8);  
S=syslin('d',A,B,C1,D12);  
[F,P]=lqr(S)  
spec(A+B*F)