Risky Strategic Interactions and the Emergence of Stationary Network Dynamics

Frank Page
Department of Economics
Indiana University
Bloomington, IN 47405
USA
fpage@indiana.edu

Joana Resende
Department of Economics and Cef.up
University of Porto
4200-464 Porto
Portugal
jresende@fep.up.pt

February 15, 2014

1 Also, Centre d’Economie de la Sorbonne, Universite Paris 1, Pantheon-Sorbonne.
2 This paper was begun by the first author during several visits to the Paris School of Economics - CES at Paris 1. The first author is grateful to CES and Paris 1 for financial support and to Cuong Le Van, Bernard Cornet, and Jean-Marc Bonnisseau for their support and hospitality during many visits to Paris. The later versions of the paper were written and the paper completed during several visits to Porto and Lisbon during the summers of 2010-2013. Both authors are especially grateful to Ehud Kalai, Matthew Jackson, A. S. Nowak, Rabah Amir, Bhaskar Dutta, Philip Reny, Gabrielle Demange, J.-J. Herings, Luca Merlino, and Jacomo Corbo for helpful discussions and comments during the writing of this paper. The second author also acknowledges financial support from FCT Research grant PTDC/EGE-ECO/115625/2009.
Abstract

We model the structure and strategy of social and economic interactions prevailing at any point in time as a directed network and we address the following open question in the theory of social and economic network formation: given rules of network formation, preferences of individuals over networks, the strategic behavior of individuals and coalitions in proposing network changes, and the micro-level riskiness of connections together with the macro-level dynamic uncertainty about bargaining outcomes over player-proposed network changes, what network and coalitional dynamics are likely to emerge and persist. Our main contributions are to formulate the problem of network and coalition formation as a dynamic, stochastic game in which uncountably many types of risky connections are possible and to show that this game possesses Nash equilibria in stationary Markov strategies (in players’ network proposal strategies), that together with the dynamic macro-level uncertainty about bargaining outcomes the players’ Nash strategies determine an equilibrium law of motion governing the process of network and coalition formation, and that this macro-level endogenous Markov process, despite there being uncountably many possible networks, generates only finitely many strategic basins of attraction, and therefore possesses only finitely many ergodic measures. Moreover, we show that in order for any specific network-coalition pair to emerge and persist in the long run, the network part of the pair must be contained in an absorbing set of each player’s equilibrium network proposal strategy (i.e., the network part must be strategically preferred), and the pair itself must be contained in an absorbing set of the equilibrium law of motion (i.e., the pair must be strategically implementable). We call any such network-coalition pair, strategically stable and dynamically consistent and we show that all such pairs reside in one of the finitely many strategic basins of attraction.

KEYWORDS: risky network connections, endogenous stationary network dynamics, dynamic stochastic games of network formation, stationary Markov equilibrium, strategic basins of attraction, Harris decomposition, ergodic probability measures.

JEL Classifications: A14, C71, C72
1 Introduction

1.1 Overview

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the “with whom”) and the strategy (“the how”) of interactions change. Our objectives here are to model the structure \textit{and} strategy of interactions prevailing at any point in time as a directed network and to analyze the co-evolution of network structure and strategic behavior by addressing the following open question in the theory of social and economic network formation: given rules of network formation, preferences of individuals over networks, the strategic behavior of individuals and coalitions in proposing network changes, and the micro-level riskiness of connections together with the macro-level dynamic uncertainty about bargaining outcomes over player-proposed network changes, what network and coalitional \textit{dynamics} are likely to emerge and persist. Thus, we propose to study the \textit{emergence of endogenous network and coalitional dynamics} resulting from the dynamic interactions of strategic behavior, network structure, and micro and macro levels of randomness in nature.

The main contributions of the paper are to formulate the problem of network and coalition formation as a dynamic, stochastic game in which uncountably many types of risky connections are possible and to show that this game possesses Nash equilibria in stationary Markov strategies (in players’ network proposal strategies), that together with the dynamic macro-level uncertainty about bargaining outcomes, the players’ Markov strategies determine an equilibrium law of motion governing the process of network and coalition formation, and that this macro-level endogenous Markov process, despite there being uncountably many possible networks, generates only \textit{finitely} many strategic basins of attraction, and therefore possesses only \textit{finitely} many ergodic measures. Moreover, we show that in order for any specific networks-coalition pair to emerge and persist in the long run, the networks part of the pair must be contained in an absorbing set of each player’s equilibrium network proposal strategy (i.e., the networks part must be strategically preferred), and the pair itself must be contained in an absorbing set of the equilibrium law of motion (i.e., the pair must be strategically implementable). We call any such networks-coalition pair, strategically stable and dynamically consistent and we show that all such pairs reside in one of the \textit{finitely} many strategic basins of attraction.

In prior work on the co-evolution of network structure and strategic behavior using static abstract game formulation games, Page and Wooders (2009a), showed that, given the rules of network formation and the preferences of individuals, these games also possess \textit{strategic basins of attraction} and these basins contain all networks that are likely to emerge and persist as the game unfolds. Moreover, Page and Wooders showed that when any one of these strategic basins contains only one network, then that network is stable against all coalitional network deviation strategies - and thus the game has a nonempty \textit{path dominance core}. Finally, Page and Wooders showed that depending on how the rules of network formation and the dominance relation over networks were specialized (via additional assumptions), any network contained in the path dominance core was pairwise stable (Jackson-Wolinsky, 1996), strongly stable (Jackson-van den Nouweland, 2005), Nash (Bala-Goyal, 2000), or consistent (Chwe, 1994).

Here we will show that there are many parallels between the static abstract game formulation and the Page-Wooders results for static games and the results obtained here for the Markovian dynamic game formulation. In particular, while Page-Wooders (2009a) show, in an abstract game setting, that there are endogenously determined collections of networks that emerge and persist (where each such network resides in one of finitely
many strategic basins of attraction), here we show, in a dynamic Markovian game setting, that there is an endogenously determined underlying stationary Markov process of network and coalition formation that emerges and persists and generates these finitely many strategic basins of attraction. This is suggested already by the seminal paper by Jackson and Watts (2002) on the evolution of networks. Jackson and Watts present to our knowledge the first theory of stochastic dynamic network formation over a finite set of linking networks governed by a Markov chain generated by the myopic strategic behavior of players (following the Jackson-Wolinsky rules of network formation) and randomness in nature. Their model builds on the earlier, nonstochastic model of dynamic network formation due to Watts (2001) - as far as we know, the first model of network dynamics (see also Skyrms and Pemantle, 2000). By considering an exogenous sequence of perturbed, irreducible and aperiodic Markov chains (i.e., each chain with a unique invariant measure) converging to the original Markov chain, they show that any pairwise stable network is necessarily contained in the support of an invariant measure - that is, in the support of a probability measure that places all its mass on sets of networks likely to form in the long run. We show here that analogous conclusions can be reached for directed networks with uncountably many connection types governed by an endogenously determined stationary Markov process of network and coalition formation resulting from the farsighted strategic behavior of players in forming networks in an environment of risky network connections and bargaining uncertainty.

In any reasonable dynamic stochastic model of network formation, it should be the case that the Markov process of network and coalition formation endogenously determined by a Nash equilibrium possesses ergodic probability measures and generate basins of attraction. It is shown here, in fact, that the endogenous Markov process possesses only finitely many ergodic measures and generates only finitely many basins of attraction - this despite there being uncountably many possible networks. This endogenous finiteness property of basins in equilibrium has serious implications for empirical work on networks. In particular, since nature does not afford the empirical observer multiple observations across states but rather only multiple observations across time, the fact that only finitely many long run equilibrium sets are possible, and more importantly, the fact that on these sets (i.e., on these basins of attraction) state averages are equal to time averages gives meaning and significance to time series observations which seek to infer the long run equilibrium network. Moreover, to the extent that networks can truly represent various social and economic interactions, our understanding of how and why the network formation process moves toward or away from any particular basin can potentially shed new light on the persistence or transience of many social and economic conditions. For example, how and why does a particular path of entrepreneurial and scientific interactions carry an economy beyond a tipping point and onto a path of economic growth driven by a particular industry - and why might it fail to do so? How and why does a particular path of product line-nonlinear pricing schedule configurations lead a strategically competitive industry to become more concentrated - or fade? These are some of the applied questions which hopefully can be addressed using a model of endogenous network dynamics.

1.2 Endogenous Stationary Network Dynamics

The approach taken here to endogenous dynamics is motivated by the observation that the stochastic process governing network and coalition formation through time is determined not only by randomness in nature through time - as envisioned in random graph theoretic approaches - but also by the strategic behavior of individuals and coalitions through time in attempting to influence the networks and coalitions that emerge under the prevailing rules of network formation and the randomness in nature. Thus, here we develop a theory
of endogenous network and coalitional dynamics that brings together elements of random graph theory and game theory in a dynamic stochastic game model of network and coalition formation. While dynamic stochastic games have been used elsewhere in economics (see, for example, Amir, 1991, 1996; Amir and Lambson, 2003; and Chakrabarti, 2008; Duffie, Geanakoplos, Mas-Colell, and McLennan, 1994; Mertens and Parthasarathy 1987, 1991; Herings and Peeters, 2004; Nowak, 2003, 2007), their application to the analysis of the evolution of social and economic networks and the emergence of equilibrium dynamics is new.

The analysis has two parts. In part (1) a discounted stochastic game model of risky network and coalition formation is constructed. We then show that with risky strategic connections and dynamic bargaining uncertainty, our stochastic, dynamic game of network formation has a stationary Markov equilibrium in player network proposal strategies. In part (2), the stability properties of the endogenous Markov process of risky network and coalition formation induced by this stationary Markov equilibrium are analyzed in detail.

The existence result presented in part (1) is based on our earlier work on the existence of stationary Markov equilibria for approximable discounted stochastic games. Here the dynamic game of network and coalition formation is formulated in a compact metric space of directed networks, possibly containing uncountably many networks, and the existence of a stationary Markov equilibrium in players’ network and coalition formation strategies is established. In addition to the uncertainty about the bargaining process over the implementation of competing network proposals, we assume that at the micro level, bilateral connections are risky. Thus, at the macro level, the state space of networks includes a risky state whose probabilistic behavior is governed by conditional density absolutely continuous with respect to a nonatomic probability measure. In Page and Resende (2013) all such risky discounted stochastic games are shown to be approximable - and therefore, are shown to possess stationary Markov equilibria. Here, in a risky discounted stochastic game of network and coalition formation consisting of \( m \) players, we show that the farsighted strategic behavior of players in attempting to influence the path of network and coalition formation generates a stationary equilibrium Markov processes of risky network and coalition formation. Thus, one of the main contributions of the paper is to provide a possible theoretical foundation in strategic behavior for the random graph theoretic approach to dynamic social and economic network formation found in the literature.

The assumptions of our discounted stochastic game model of network formation are similar to those required to establish the existence of stationary correlated equilibria in discounted stochastic games (e.g., Nowak and Raghavan 1992) and subgame perfect equilibria in discounted stochastic games (e.g., Mertens and Parthasarathy 1987, Salon 1998, and Maitra and Sudderth 2007). Our model has six primitives consisting of the following: (i) a feasible set of directed networks representing all possible configurations of social or economic interactions, (ii) a feasible set of coalitions allowed to form under the rules of network formation for the purpose of proposing alternative networks, (iii) a state space consisting of feasible pairs of regular states and risky states, where each regular state is given by a network-coalition pair and each noisy state is given by a realized network that

---

1Within the context of a stochastic game model similar to model analyzed by Mertens and Parthasarathy (1987) and Nowak and Raghavan (1992), Page and Resende (2014) establish a general existence result for stationary Markov Nash equilibria for a large class of discounted stochastic games consisting all approximable discounted stochastic games. While the existence of Nash equilibria in stationary Markov strategies for discounted stochastic games with finite or countable state spaces and compact metric action spaces has long been established (e.g., see Federgruen, 1978), the existence of such equilibria for discounted stochastic games with uncountable state spaces and compact metric action spaces has been an open question since such games were first studied by Himmelberg, Parthasarathy, Raghavan, and Van Vleck (1976).
differs from the regular network by noise realized during implementation of the regular network that was stochastically chosen by the bargaining process, (iv) a set of players and player constraint correspondences specifying for each player, in each state (consisting of a regular state, risky state pair) the set of feasible alternative networks that a player can propose under the rules of network formation as a member of the current or status quo coalition - and as a nonmember, (v) a set of player discount rates and payoff functions defined on the graph of players’ constraint correspondence, and (vi) a stochastic law of motion consisting of a law governing the regular state transitions and a conditional probability measure over the feasible noisy states conditional on the regular state. This stochastic law of motion represents nature and specifies the probability with which each possible new status quo network-coalition (i.e., new state) might emerge as a function of the status quo networks-coalition pair (i.e., the current state) and bargaining, the profile of player-proposed new status quo networks (i.e., the current action profile). Using these primitives, we construct a discounted stochastic game model of network formation and show that it possesses a stationary Markov equilibrium in network proposal strategies. More importantly, we are able to conclude via classical results due to Blackwell (1965) (also see Himmelberg, Parthasarathy, and vanVleck (1976)), Nowak and Raghavan (1992), and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)) that this equilibrium over stationary Markov strategies is optimal against player defections to other network proposal strategies (including history-dependent proposal strategies) - thus showing that our decision to focus on stationary strategies (i.e., strategies that depend only on the status quo network-coalition pair) is well-founded.

In part (2), we analyze the stability properties of the endogenous stationary Markov process of network and coalition formation. In particular, using methods of stability analysis essentially due to Nummelin (1984) and Meyn and Tweedie (2009) - and based on the profound work of Doeblin (1937, 1940) - we show that the equilibrium Markov process of network and coalition formation possesses ergodic probability measures and generates strategic basins of attraction.\(^2\) We then study in some detail the number and structure of these basins of attraction as well as the structure of the set of invariant probability measures. More importantly, we show that, in a state space with uncountably many networks, the equilibrium process possesses only finitely many ergodic measures and basins of attraction. Also, in part (2) we will introduce the notions of strategic stability and dynamic consistency and using these notions extend the definition of network stability to the dynamic Markov setting developed here. We will then show that states that are stable must necessarily reside in the strategic basins of attraction generated by the endogenous network dynamic. In coming to these conclusions, it will become clear that in dynamic Markov games of network and coalition formation, stability has two masters: strategic behavior and the laws of nature - with the laws of nature being dominate.

1.3 Related Literature

To our knowledge, the first paper to study endogenous dynamics in a related model is the paper by Konishi and Ray (2003) on dynamic coalition formation. The primitives of their model consist of (i) a finite set of outcomes (possibly a finite set of networks), (ii) a set of coalitional constraint correspondences specifying for each coalition and each status quo outcome, the set of new outcomes a coalition might bring about if allowed to do so, and (iii) a discount rate and set of player payoff functions defined on the set of all

\(^2\)Our stability results for equilibrium Markov processes of network and coalition formation, while classical in form and appearance, are completely new. Unlike in the classical setting where the state space is finite or countable, the state space here is uncountable, consisting of uncountably many networks and coalitions pairs.
outcomes. Konishi and Ray show that their model possesses a stochastic law of motion governing movement from one outcome to another and a consistent valuation function such that (a) if a move from one outcome to another takes place with positive probability, then for some coalition this move makes sense in that no coalition member is made worse off by the move and no further move makes all coalition members better off, and (b) if for a given outcome there is another outcome making all members of some coalition better off and no further outcome makes this coalition even better off, then a move to another outcome takes place with probability 1 (i.e., the probability of standing still at the given outcome is zero). Stated loosely, then, Konishi and Ray show that for their model there exists a law of motion which generates co-ally improving moves from one outcome to another (i.e., in our case it would be from one network to another).

Our model differs from the model of Konishi and Ray in several respects. First, in our model movements from one network (outcome) to another are largely determined by the strategic behavior of individuals. In our model, equilibrium strategic behavior, together with riskiness, are central to determining equilibrium network dynamics.

Second, whereas Konishi and Ray, for technical reasons, restrict attention to a finite set of outcomes (in our model, a finite set of networks), we allow for uncountably many networks - this to allow for consideration of networks with a large number of nodes or networks with uncountably many arc types. This is more than a technical nicety. In order to capture the myriad and potentially complex nature of interactions between players (say for example in a stock market or in a contracting game with multiple principals and multiple agents) we must allow there to be uncountably many possible types of interactions. In our model the set of potential interactions are represented by a set of arc types (in fact, by a compact metric space of arc types) with each arc type (or arc label) representing a particular type of interaction (or connection) between nodes in a directed network. Thus, because we allow for uncountably many arc types in describing the interactions between nodes, in our model there are uncountably many possible networks (or outcomes, in the language of Konishi and Ray). Moreover, in order to model large networks (i.e., networks with many nodes), in our model we can allow there to be infinitely many nodes - although here we focus exclusively on the finite nodes case. Third, while Konishi and Ray restrict attention at the outset to Markov laws of motion, we will show that our strategically determined equilibrium Markov process of network and coalition formation is robust against all possible alternative dynamics, even those induced by history-dependent types of strategic behavior. Thus, at least for the class of Konishi-Ray types of models, we will show that Markov laws of motion are stable and robust with respect to other forms of history-dependent laws of motion.3

Finally, we take the rules of network formation as given primitives of our model. We show that the interactions of strategic behavior, network structure, and stochastically controllable randomness of nature generate an equilibrium process of network change and coalition formation consistent with these rules. We also show that this endogenously determined process possesses a nonempty set of ergodic measures and generates strategic basins of attraction. In Konishi-Ray, there are no rules of coalition formation – rules specifying how the process moves from one state to another; instead they focus on transitions consistent with improvement properties for coalitions.

In contrast to Konishi-Ray, Dutta, Ghosal, and Ray (2005) consider strategic behavior in a dynamic game of network formation over a finite set of undirected linking networks (rather than directed networks) under a particular set of network formation rules. They show existence of a Nash equilibrium and identify conditions under which efficiency can be

---

3By a Markov law of motion we mean a stochastic law of motion where probabilistic movements from one outcome or network to another depend only on the current outcome rather than on some history of outcomes.
sustained in equilibrium - thus, continuing in a dynamic setting the seminal work of Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) on equilibrium and efficiency. Here our focus is on equilibrium and stability rather than equilibrium and efficiency and our analysis is carried out in a dynamic, stochastic game model of network and coalition formation, admitting all forms of network formation rules, over an uncountable set of directed networks. Dutta et al. (2005) restrict attention to Markov network formation strategies and show that there is an equilibrium in this class. In contrast, we show for the class of all strategies that there is an equilibrium in stationary Markov strategies; and therefore, by Blackwell’s classical result (Blackwell, 1965, Theorem 6f) we conclude that this type of equilibrium is robust against defections by individual players to any other type of strategy. Moreover, as mentioned above, we show that in general, the resulting equilibrium Markov process of network and coalitional formation possesses finitely many ergodic measures and generates finitely many network and coalitional basins of attraction.

We view the starting point of our research to be the pioneering work of Jackson and Watts (2002) already discussed briefly above. Our model of endogenous network and coalitional dynamics extends their work on stochastic network dynamics in several respects. First, in our model players behave farsightedly in attempting to influence the path of network and coalition formation - farsighted in the sense of dynamic programming (e.g., Dutta, Ghosal, and Ray (2005))⁴. Moreover, in our model the game is played over a (possibly) uncountable collection of directed networks under general rules of network formation which include not only the Jackson-Wolinsky rules, but also other more complex rules. In our model the law of motion is such that the randomness of nature is Markovian a stochastically controllable, rather than uncontrollable and i.i.d. as in Jackson and Watt. Extending the notion of stability to a dynamic setting, one of the benchmarks for our research is to show that in a Markov model of network and coalition formation, if a state is dynamically stable, then in order to persist, it must be contained in one of finitely many strategic basins of attraction, and therefore, contained in the support of an ergodic probability measure.

2 Networks

2.1 Riskless Strategic Connections and Networks

The basic ingredients of our network model are as follows:

[A-1] (nodes, arcs, and players)

\[ N = \text{a finite set of nodes with typical elements } i \text{ and } j, \text{ equipped with the discrete metric } d_N; \]

\[ A = \text{a closed subset of a compact metric space, } (E,d_E), \text{ of arc types, with typical element } a, \]

\[ D = \text{a finite set of players, with typical element } d. \]

Let \( P(N \times N) \) denote the collection of all nonempty subsets of \( N \times N \), with typical element \((i, j)\) called a node pair or a pre-connection. We will refer to any nonempty subset of pre-connections, \( C \subseteq P(N \times N) \), as a pre-network. In order for a pre-connection to become a connection, it must be assigned a connection type. The set of arcs, \( A \), provides us with the set of connection types.

We begin by defining the notion of a directed, strategic connection.

**Definition 1 (Riskless Strategic Connections)**

Given node set \( N \), arc set \( A \), a riskless strategic connection is an ordered pair \((a,(i,j))\) \( \in A \times (N \times N) \) consisting of an arc type \( a \in A \) and an order pair of nodes, \((i, j)\) \( \in N \times N \).

Thus, a connection, \((a,(i,j))\), is a pre-connection, \((i, j)\), together with a arc label, \( a \), indicating that nodes \( i \) and \( j \) are connected by an arc type \( a \) from node \( i \) to node \( j \). The set of all feasible directed connections is given by

\[ K := A \times (N \times N). \]

Given our network assumptions [A-1], the set of all possible directed connections, \( K \), is a compact metric space with sum metric

\[ d_K((a,(i,j)),(a',(i',j'))):=d_E(a,a') + d_N(i,i') + d_N(j,j'). \]

Because the set of all feasible directed connections, \( K \), is a compact metric space, the hyperspace of nonempty, closed subsets of directed connections, \( P_f(K) \), equipped with the Hausdorff metric, \( h \), is also a compact metric space. Recall that the Hausdorff metric, \( h \), induced by the metric \( d_K \) on the feasible set of directed connections is given by

\[ h(G,G') := \max \{e(G,G'),e(G',G)\} \]

where \( e(G,G') \) is the excess of \( G \in P_f(K) \) over \( G' \in P_f(K) \) given by

\[ e(G,G') := \max_{(a,(i,j)) \in G} \text{dist}((a,(i,j)),G') \]

and \( \text{dist}((a,(i,j)),G') \) is the distance from the connection \((a,(i,j)) \) \( \in G \) to the nonempty, closed subset of connections \( G' \), given by

\[ \text{dist}((a,(i,j)),G') := \inf_{(a',(i',j')) \in G'} d_K((a,(i,j)),(a',(i',j'))). \]

\[ d_N(i,j) := \begin{cases} 
1 & \text{if } i \neq j \\
0 & \text{if } i = j.
\end{cases} \]

5Under the discrete metric the distance between two nodes \( i \) and \( j \) in \( N \) is given by
In general, given any metric space \((X,d_X)\) we will denote by \(P_{d_X}(X)\) the collection of all nonempty, closed subsets of \(X\), and where no confusion is possible concerning the metric on \(X\), we will simply denote this collection by \(P_f(X)\) and we will often refer to \(P_f(X)\) as a hyperspace.

A directed network is defined as follows:

**Definition 2 (Riskless Networks)**

Given node set \(N\), arc set \(A\), a riskless network, \(G\), is a nonempty, closed subset of the set riskless strategic connections, \(K\).

The hyperspace of all directed networks is given by \(P_f(K)\).

Under our definition of a directed network, loops are allowed - a loop being a connection where an arc goes from a given node back to that given node.\(^6\) Also, under our definition an arc can be used multiple times in a given network and multiple arcs (even uncountably many) can go from one node to another. However, under our definition no arc \(a\) is allowed to go from a node \(i\) to a node \(j\) multiple times.

The following notation is useful in describing networks. Given directed network \(G \in P_f(K)\), let

\[
\begin{align*}
G(a) & := \{(i,j) \in N \times N : (a,(i,j)) \in G\}, \\
G(ij) & := \{a \in A : (a,(i,j)) \in G\}.
\end{align*}
\]

In network \(G\),

- \(G(a)\) is the set of node pairs connected by arc \(a\), and
- \(G(ij)\) is the set of arcs from node \(i\) to node \(j\).

The set \(G(a)\) is called the section of network \(G\) at arc \(a\), while \(G(ij)\) is called the section of network \(G\) at pre-connection \((i,j)\). If for some arc \(a \in A\), \(G(a)\) is empty, then arc \(a\) is not used in network \(G\). Also, if for some node \(i \in N\), \(G(ij)\) and \(G(ji)\) are empty for all \(j \neq i\), then node \(i\) is isolated.

### 2.2 Networks as Functions

Note that each network \(G \in P_f(K)\) induces a unique arc mapping,

\[
(i,j) \mapsto G(ij) \in P_f(A),
\]

defined on some subset of the set of pre-connections, called the domain of \(G\), taking values in the hyperspace of nonempty, closed subsets of arcs. Formally the domain, \(D(G)\), and the range, \(R(G)\), of the arc mapping induced by network \(G \in P_f(K)\) are given by

\[
\begin{align*}
D(G) & := \{(i,j) \in N \times N : G(ij) \neq \emptyset\}, \\
R(G) & := \{E \in P_f(A) : G(ij) = E \text{ for some } (i,j) \in N \times N\}.
\end{align*}
\]

Conversely, any set-valued mapping, \((i,j) \mapsto G(ij)\), defined on a subset of pre-connections, \(C \subseteq N \times N\), taking values in \(P_f(A)\) uniquely identifies a directed network via the graph of the mapping,

\[
GrG := \{(a,(i,j)) \in K : a \in G(ij)\}
\]

\(^6\)By allowing loops we are able to represent a network having no connections between distinct nodes as a network consisting entirely of loops at each node.
Denote by
\[ U(C, P_f(A)) \] (7)
the collection of all arc mappings, \( G(\cdot) \), with domain \( C \subseteq N \times N \) (a pre-network) and range contained in \( P_f(A) \). Thus, we have

(i) for directed network \( G \in P_f(K) \), the induced arc correspondence, \((i,j) \rightarrow G(ij)\), is contained in \( U(C, P_f(A)) \), and

(ii) for arc mapping, \((i,j) \rightarrow G(ij)\), contained in \( U(C, P_f(A)) \), \( GrG \) is contained in \( P_f(K) \).

We will denote by \( |E| \) the cardinality of any set \( E \) (i.e., the number of elements in \( E \)), with the convention that \( |E| = 0 \) if and only if \( E = \emptyset \). We will also adopt the notational convention that \( P^n(E) \) denotes the collection of all nonempty subsets of \( E \) consisting of no more than \( n \) elements, while \( P^{(n)}(E) \) will denote the collection of all nonempty subsets of \( E \) consisting of exactly \( n \) elements. Thus, if \( S \in P^n(E) \), then \( 1 \leq |S| \leq n \) and if \( S \in P^{(n)}(E) \), then \( |S| = n \).

Note that network \( G \in P_f(K) \) in addition to inducing a unique arc mapping, \((i,j) \rightarrow G(ij)\), also induces a unique cardinality mapping
\[(i,j) \rightarrow |G(ij)| \in \{0, 1, 2, 3, \ldots\}. \tag{8} \]

Using the cardinality mapping, \((i,j) \rightarrow |G(ij)|\), we can compute the out degree and the in degree of any node in the network. In particular, for network \( G \in P_f(K) \) with cardinality mapping \( |G(\cdot)| \), the out degree of node \( i \) in network \( G \) is given by \( \sum_{j \in N} |G(ij)| \), while the in degree of node \( i \) is given by \( \sum_{j \in N} |G(ji)| \).

### 2.3 Networks as Arc Selections

#### 2.3.1 Arc Categories, Unique Connections, and the Feasible Arc Mapping

A pre-connection may be a connection in several different networks, where each such network is distinguished by a specific category of connection types used in the network. For example, consider a pre-connection, \((i,j) \in N \times N\), between nodes \( i \) and \( j \) representing banks. In an interbank loan network, it is reasonable to suppose that an arc type from \( i \) to \( j \) is given by nonnegative numbers from some closed bounded interval representing possible loan amounts from bank \( i \) to bank \( j \). Thus, in an interbank loan network, \( A \) might be given by some closed bounded interval, \([0, M]\). It is also reasonable to suppose in such a network there is one and only one such connection from bank \( i \) to bank \( j \). We will call such a network a categorical unique connections network - i.e., a CUC network. Other categories of connections between banks can then be represented by other networks with unique connections - with the entire banking network being given by the union of these categorical networks each with unique connections.

#### 2.3.2 Feasible Networks as Arc Selections

Given that the set of arc types, \( A \), represents a specific category of arcs, as one of the primitives of our model we will assume that there is defined on the set of all pre-connections, \( N \times N \), a feasible arc correspondence
\[ A(\cdot) : N \times N \rightarrow P_f(A). \tag{9} \]

Thus, corresponding to each pre-connection, \((i,j) \in N \times N\), there is a nonempty set of connection types, \( A(ij) \in P_f(A) \), consisting of all the connection types possible from node \( i \) to node \( j \).
Given feasible arc correspondence, \((i, j) \rightarrow A(ij)\), we can then think of a feasible network \(G \in P_f(K)\) as specifying a domain of pre-connections, \(\mathcal{D}(G)\), such that on this domain, the network’s induced arc mapping, \((i, j) \rightarrow G(ij)\), is a selection of the feasible arc correspondence. Thus, from this perspective, given arc category \(A\), a feasible network \(G\) specifies a domain of pre-connections, \(\mathcal{D}(G)\), and a single-valued function,

\[ G(\cdot) : \mathcal{D}(G) \rightarrow A \]

such that for each \((i, j) \in \mathcal{D}(G)\),

\[ G(ij) \in A(ij) \subset A. \]

We will denote by

\[ \Sigma(\mathcal{D}(G), A(\cdot)) \]

the set of all arc selections induced by network \(G \in P_f(K)\).

We will take as our feasible set of networks, \(\mathcal{G} \subset P_f(K)\), the \(h\)-closed subcollection of CUC networks that induce arc selections of the feasible arc correspondence, \(A(\cdot)\). Formally, this subcollection is given by,

\[ \mathcal{G} := \{ G \in P_f(K) : \forall (i, j) \in \mathcal{D}(G), |G(ij)| = 1 \text{ and } G(ij) \in A(ij) \}. \] (11)

Conversely, given any pre-network, \(C \in P(N \times N)\), we will denote the collection of all selections of the feasible arc correspondence, \(A(\cdot)\), with domain \(C\) by

\[ \Sigma(C, A(\cdot)), \]

and we will denote by \(G(\cdot) : C \rightarrow A\) a typical function contained in \(\Sigma(C, A(\cdot))\). Thus,

\[ G(\cdot) \in \Sigma(C, A(\cdot)) \text{ if and only if } G(ij) \in A(ij) \text{ for all } (i, j) \in C, \]

and we say that the function \(G(\cdot)\) is a selection from the feasible arc correspondence, \(A(\cdot)\), for pre-network \(C\).

Viewing feasible directed networks as single-valued arc selections from feasible the arc correspondence, \(A(\cdot)\), and letting \(\Sigma(C, A(\cdot))\) denote the collection of all such selections given \(C\), the set of all possible feasible directed networks can be written as the union of all such selections over the collection of all possible pre-networks. Thus, we have

\[ \mathcal{G} = \bigcup_{C \in P(N \times N)} \Sigma(C, A(\cdot)), \]

and because the set of nodes is finite, the set of all possible pre-networks, \(P(N \times N)\), is finite.

### 2.3.3 Domain Equivalence Classes

Given any feasible set of networks \(\mathcal{G}\) and given any network \(G \in \mathcal{G}\), we say that another network \(G' \in \mathcal{G}\) is in the same domain equivalence class with \(G\) if networks \(G\) and \(G'\) have the same domain - i.e., \(\mathcal{D}(G') = \mathcal{D}(G)\). Formally, the equivalence class determined by feasible network \(G \in \mathcal{G}\) is given by

\[ \mathcal{G}_\mathcal{D}(G) := \{ G' \in \mathcal{G} : \mathcal{D}(G') = \mathcal{D}(G) \}. \] (14)

Alternatively, for each pre-network, \(C \in P(N \times N)\), the collection of networks given by

\[ \mathcal{G}_C := \{ G \in \mathcal{G} : \mathcal{D}(G) = C \} \]

(15)
forms the equivalence class of all networks in $\mathcal{G}$ having domain (or pre-network) $C$. Note that if $C \neq C'$ then $\mathcal{G}_C \cap \mathcal{G}_{C'} = \emptyset$ and $\mathcal{G} := \bigcup_{C \in \mathcal{P}(N \times N)} \mathcal{G}_C$. Thus, the collection

$$\{\mathcal{G}_C : C \in \mathcal{P}(N \times N)\} \quad (16)$$

is a partition of the space of unique connections networks into domain equivalence classes. We will return to this unique decomposition of feasible networks into domain equivalence classes in our discussion below of nonatomic probability spaces of networks.

2.4 Measuring the Distance Between Directed Networks

In order to analyze the co-evolution of strategic behavior, network structure and equilibrium dynamics, we must find a topology for the space of directed networks that is simultaneously coarse enough to guarantee compactness and fine enough to discriminate between differences across networks that are due to differences in the ways nodes are connected (via differing arc types) and differences across networks that are due to the complete absence of connections. Here we resolve this topological dilemma by equipping the space of directed networks, $P_f(K)$, with the Hausdorff metric $h$. Because the set of directed connections, $K := A \times (N \times N)$, is a compact metric space, the space of directed networks, $P_f(K)$ equipped with the Hausdorff metric is automatically compact (see Section 7 below, also see Section B.11 in Hildenbrand 1974, or Sections 3.16-3.18 in Aliprantis and Border 2006). Moreover, given the nature of the discrete metric on the set of nodes, it is easy to show that if the Hausdorff distance between any pair of networks $G$ and $G'$ is less than $\varepsilon \in (0, 1)$, then the networks can differ only in the ways a given set of node pairs are connected - and not in the set of node pairs that are connected. In particular, if for networks $G$ and $G'$, $h(G, G') < \varepsilon < 1$, then $\mathcal{D}(G) = \mathcal{D}(G')$ and we have

$$(a, (i, i')) \in G \text{ if and only if } (a', (i, i')) \in G'$$

for arcs $a$ and $a'$ with $d_A(a, a') < \varepsilon$. Thus, for $h(G, G')$ sufficiently small (less than 1), $G$ and $G'$ are in the same pre-network equivalence class (i.e., in order for two networks to be close, they must be in the same domain equivalence class).

To illustrate the sensitivity of the Hausdorff metric topology to absence or presence of connections across networks, consider the following example. Suppose that the set of nodes is given by $N := \{i_1, i_2, i_3\}$, while the set of arcs types is given by $A = [0, 1]$. We can think of arc types $a \in [0, 1]$ as representing intensity levels or flow levels from one node to another or as expressing the probabilities with which one node interacts with
Consider the three networks, \( G_1, G_2, \) and \( G_3 \) depicted in Figure 1.

Note that while the domains of networks \( G_1 \) and \( G_2 \) contain the node pair \((i_1, i_2)\), the domain of network \( G_3 \) does not. From the perspective of the cardinality functions, note also that while

\[ |G_1(i_1i_2)| = |G_2(i_1i_2)| = 1, \]

\[ |G_3(i_1i_2)| = 0. \]

In network \( G_1 \) the connection from \( i_1 \) to \( i_2 \) is inactive (i.e., has a zero intensity level), that is, \((0, (i_1, i_2)) \in G_1\). In network \( G_2 \) the connection from \( i_1 \) to \( i_2 \) is weak, that is, \((.001, (i_1, i_2)) \in G_2\). However, in network \( G_3 \), there is no connection at all from \( i_1 \) to \( i_2 \). Under the network metric \( h_K \) (see 3), networks \( G_1 \) and \( G_2 \) are close, while networks \( G_1 \) and \( G_3 \) as well as networks \( G_2 \) and \( G_3 \) are far apart. In particular, \( h_K(G_1, G_2) = .001 \), while

\[ h(G_1, G_3) = 2 \]

and

\[ h(G_2, G_3) = 2 - .001. \]

In the context of linking networks, this class of networks (i.e., networks with constrained, variable link strength) has recently been used to investigate the endogenous formation of efficient and reliable communications networks by Bloch and Dutta (2009). See Page and Wooders (2009b) for a further discussion of differences between linking networks with variable length strength and directed networks with heterogeneous arc types.
In the analysis to follow, one of our main objectives will be to better understand the emergence and stability properties of equilibrium network dynamics generated by the endogenous interplay between network structure and strategic behavior in the formation of networks over time. In order to achieve this objective, we must allow for the emergence of networks where some connections are absent altogether (i.e., where some node pairs are not connected in any direction by any arc types, as in network \( G_3 \) in Figure 1). The Hausdorff metric topology on the space of networks is particularly well suited for the type of analysis required to achieve this objective.

Thus the hyperspace of networks \((P_f(K), h)\) is compact metric space and the feasible subset of unique connections networks, \(G\), is an \(h\)-closed subspace of \(P_f(K)\). We will denote by \( B(\mathbb{G}) \) the Borel \(\sigma\)-field of subsets of networks generated by the Hausdorff metric, \(h\), restricted to \(G\).

Recall that the equivalence class determined by feasible network \(G \in \mathbb{G}\) is given by

\[ \mathcal{G}_{D(G)} := \{ G' \in \mathbb{G} : D(G') = D(G) \} . \]

For each \(G \in \mathbb{G}\), it is easy to verify that \(\mathcal{G}_{D(G)}\) is \(h\)-closed. Moreover, viewing \(\mathcal{G}_{D(\cdot)}\) as a function from \(\mathbb{G}\) into the collection of domain equivalence classes, it is easy to show that \(\mathcal{G}_{D(\cdot)}\) is continuous with respect to the Hausdorff metric. Thus recalling that

\[ dist_h(G', \mathcal{G}_{D(G)}) := \inf_{G'' \in \mathcal{G}_{D(G)}} h(G', G'') \]

is the distance from feasible network \(G'\) to \(\mathcal{G}_{D(G)}\), the open enlargement about the equivalence class of networks, \(\mathcal{G}_{D(G)}\), (i.e., the open \(\varepsilon\)-ball about \(\mathcal{G}_{D(G)}\)) is given by

\[ B_h(\varepsilon, \mathcal{G}_{D(G)}) := \{ G' \in \mathbb{G} : dist_h(G', \mathcal{G}_{D(G)}) < \varepsilon \} . \]

Because \(\mathcal{G}_{D(\cdot)}\) is continuous, it is upper semicontinuous as well as lower semicontinuous. Thus for any \(\varepsilon > 0\), there is a \(\delta^\text{usc}_\varepsilon > 0\) such that if \(G' \in B_h(\delta^\text{usc}_\varepsilon, G)\), then

\[ \mathcal{G}_{D(G')} \subset B_h(\varepsilon, \mathcal{G}_{D(G)}) , \]

and there exists \(\delta^\text{lsc}_\varepsilon > 0\) such that if \(G' \in B_h(\delta^\text{lsc}_\varepsilon, G)\), then

\[ \mathcal{G}_{D(G')} \cap B_h(\varepsilon, \mathcal{G}_{D(G)}) \neq \emptyset . \]

### 2.5 Players and Coalitions

The path taken by a network through time depends in large measure on the actions taken by groups of players in attempting to influence how the network changes across time by influencing the stochastic process of network formation. Thus, coalitions play a central role in our model.

We will assume that there are \(m = |D|\) players and in our model the set of players, \(D\), is not necessarily equal to the set of nodes \(N\) (i.e., we make a distinction between players and nodes). The set \(P^n(D)\) denotes the collection of all nonempty subsets or coalitions of players, consisting of no more than \(n = 1, 2, \ldots, m\) players, with typical element \(S\), while the set \(P^{(n)}(D)\) denotes the collection of all coalitions of players, consisting of exactly \(n = 1, 2, \ldots, m\) players, again with typical element \(S\). Note that if \(n \geq m\), then \(P^n(D)\) simply denotes the collection of all nonempty subsets of players, and if \(n = m\), then \(P^{(n)}(D)\) consists of exactly one coalition, the grand coalition, and if \(n > m\), then \(P^{(n)}(D)\) consists of no coalitions - i.e., \(P^{(n)}(D)\) is empty. For \(n \geq m\), we will usually just write \(P(D)\) to denote the collection of all coalitions. Recall that \(D\) denotes the set of \(m\) players (\(|D| = m\) and \(D\) is not necessarily equal to the set of nodes \(N\)) with typical element \(d\) and
$P^n(D)$ denotes the collection of all nonempty coalitions of players, consisting of no more than $n = 1, 2, \ldots, m$ players, with typical element $S$, while $P^{(n)}(D)$ denotes the collection of all coalitions of players, consisting of exactly $n = 1, 2, \ldots, m$ players.

Often restrictions on the feasible set of coalitions are the result of the rules of network formation. Examples of feasible sets of coalitions include the set,

$$P^{(2)}(D) = \{S \in P^m(D) : |S| = 2 \},$$

where each feasible coalition consists of 2 players, the set,

$$P^2(D) = \{S \in P^m(D) : |S| \leq 2 \},$$

where each feasible coalition consists of, at most, 2 players, and the set

$$P^{(1)}(D) = \{S \in P^m(D) : |S| = 1 \},$$

where each feasible coalition consists of 1 player.

We will denote by $F \subset P^m(D)$ the feasible set of coalitions and we will equip the feasible set of coalitions with the discrete metric $d_F$ (i.e., $d_F(S', S) = 0$ if $S' = S$, and $d_F(S', S) = 1$ if $S' \neq S$).

## 3 Risky Networks

### 3.1 Risky Strategic Connections

Our focus will be on discounted stochastic games of network formation over risky networks. All games have as their basic ingredients a set of players each with a strategy set and a payoff function over strategy profiles. In a discounted stochastic game, there is in addition, a probability space of states to model risk. Here we will assume that there are two types of risk: bargaining risk and connection risk. In addition, in a discounted stochastic game there is a law of motion to model how these risks might unfold across time in response to players’ efforts to control and influence the path these risks take through time. Here, in addition to these partially controllable risks, we want to allow for risks in network formation that are only indirectly controllable - or not controllable at all. Thus, the state space and the law of motion, in addition to providing a model of partially controllable risks, must also provide enough structure to model the presence of indirectly controllable, or uncontrollable risks in (or connection shocks to) strategic interactions leading to the formation of the network.

A risky strategic interaction (or a risky connection) can be described as follows: In any riskless network, $G \in G$, a typical connection is given by an ordered pair, $(a, (i, j))$, denoting a connection of type $a \in A(ij)$ from node $i$ to node $j$ in network $G$. Such a connection looks like the following:

![Figure 2](image)

Our objective is to take into account, at the macro level of the entire network, the possibility that, at the micro level of individual connections, the final connection that emerges during the game of network formation might instead look like the following:

![Figure 3](image)
where the final arc type, $\tilde{a}$, in Figure 3 describing the realized connection between nodes $i$ and $j$ differs from the intended arc type, $a$, from $i$ to $j$ in Figure 2 by noise.

To accomplish this, we begin by assuming that the state space $\Omega$ is given by a set of $3$-tuples,

$$\Omega := \left\{ (\tilde{G}, (G, S)) \in G \times (G \times F) : \tilde{G} \in \mathcal{G}_{D(G)} \right\},$$  

(17)

where $(G, S)$ is the regular state - a network-coalition pair - and $\tilde{G}$ is the risky state - a network determined by noise and in the same domain equivalence class, $\mathcal{G}_{D(G)}$, as $G$. In state $\omega := (\tilde{G}, (G, S)) \in \Omega$, a typical connection in network $G \in \mathcal{G}$ is given by $(a, (i, j))$ - as in Figure 2 - while in network $\tilde{G} \in \mathcal{G}_{D(G)}$, a typical connection (with the same underlying pre-connection, $(i, j)$) is given by $(\tilde{a}, (i, j))$ - as in Figure 3. Thus, the micro-level connection risk, represented by the typical connection $(\tilde{a}, (i, j))$, at the macro-level is represented by the risky network $\tilde{G}$.

Regular states, $(G, S)$, are distributed according to a probability measure,

$$\rho(d(G', S')|\omega, (G, S)), G_D),$$

conditioned by the current state, $\omega := (\tilde{G}, (G, S))$, and the $m$-tuple of networks, $G_D := (G_1, \ldots, G_m)$, proposed the players for the coming period. We assume that for all pairs of current states and player proposal $m$-tuples, $(\omega, G_D)$, $\rho(d(G', S')|\omega, G_D)$ is absolutely continuous with respect to a fixed product probability measure, $\mu(d(G, S)) := \nu(dG) \times \eta(dS)$, defined on $\mathcal{G} \times \mathcal{F}$. In our model, the stochastic kernel,

$$G_D \rightarrow \rho(d(G', S')|\omega, G_D) \in \mathcal{P}(\mathcal{G} \times \mathcal{F})$$

represents the stochastic behavior of the bargaining process by which players in current state $\omega \in \Omega$, whose $m$-tuple of network proposals is $G_D$, reach an agreement about which network, say $G'$, to implement next and which coalition, $S'$, should be next to propose substantive changes in the coming network, $G'$.

Thus, the uncertainty about the coming regular state, $(G', S')$, given current or status quo state

$$\omega := (\tilde{G}, (G, S)) \in \Omega$$

and player proposal $m$-tuple

$$G_D := (G_1, \ldots, G_m) \in \Phi(\omega),$$

governed by the conditional probability measure, $\rho(d(G', S')|\omega, G_D)$, is due to the stochastic nature of bargaining process and is the main source of the dynamic macro-level uncertainty.

In order to model the distinction between players, $d \in S$, whose turn it is to move and other players, $d \notin S$, we will assume that the only proposal available to players not in coalition $S$ is the current status quo network (i.e., these players cannot propose substantive changes in the current network $G$) - while players $d \in S$ can propose substantive changes to the current network in accordance with the rules of network formation. Thus, all players propose but only players in $S$ can propose substantive changes - and thus players, $d \in S$, have an opportunity to at least partially control the random path of the coming

---

*In most models in the literature, the coalition $S$ whose turn it is to move is usually a set consisting of two players with each player pair being chosen by an i.i.d. process - i.e., player pairs are chosen in each period from some fixed probability measure over player pairs. Here, we are assuming that players $S$ whose turn it is to move are chosen via a partially controllable, stochastic process, rather than by an i.i.d. process.*
regular state, \((G', S')\), via their network proposals through the stochastic kernel, \(G_S \rightarrow \rho(d(G', S')|\omega, (G_S, G_{-S}))\).

While regular states are the result a partially controllable stochastic bargaining process, risky states, \(\tilde{G}\), are the result of only indirectly controllable noise in the implementation process. In particular, we assume that risky states, \(\tilde{G}\), are chosen according to the probability measure,

\[ \varepsilon(d\tilde{G}|G, S), \]

conditioned by the regular state, \((G, S)\), and absolutely continuous with respect to a fixed nonatomic probability measure, \(\lambda(d\tilde{G})\), defined on \(\mathcal{G}_D(G)\). By assuming that the risk-determined network \(\tilde{G} \in \mathcal{G}_D(G)\) and the penultimate network \(G\) are contained in the same domain equivalence class - and therefore that the final network \(\tilde{G}\) and the penultimate network \(G\) differ only in the arcs used in connecting node pairs in the identical domains \(\mathcal{D}(\tilde{G})\) and \(\mathcal{D}(G)\) - we are assuming that the riskiness in the network formation process is not sufficiently strong to alter the pre-network (i.e., is not sufficiently strong to break or create connections in the penultimate network \(G\)), but is sufficiently strong to alter, in random ways, the exact types of the connections determined initially during the process of network bargaining (see Figures 2 and 3 above).

In summary, for any coming state \(\omega' := (G', (G', S'))\), \(\tilde{G}' \in \mathcal{G}_D(G)\) is the final network, a result of the resolution of the riskiness generated during the attempt to put into place the intended network, \(G'\) - where the intended network, \(G'\), is the result of players in status quo state, \(\omega := (\tilde{G}, (G, S))\), bargaining over an \(m\)-tuple, \(G_D\) of network proposals and the resolution of randomness in the bargaining process. Note that in moving from the status quo, \(\omega := (\tilde{G}, (G, S))\), to the coming state \(\omega' := (\tilde{G}', (G', S'))\) the coalition \(S'\) of players whose turn it will be to move next is not altered by the riskiness in connections (by the noise risk). Thus, in our model, at each time point, each player \(d\) has an opportunity to propose a network in accordance with the rules of network formation and to bargain with other players in order to try to influence or control the stochastic process of network and coalition formation. At each time point, each player \(d\) can propose any network, \(G_d\), contained in the player’s state-contingent constraint set,

\[ \Phi_d(\omega) = \Phi_d(\tilde{G}, (G, S)), \]

but for players \(d\) not members of the coalition \(S\), \(\Phi_d(\tilde{G}, (G, S)) = \{\tilde{G}\}\). Thus, if the current state is \(\omega := (\tilde{G}, (G, S))\) and \(G_D := (G_d)_{d \in D}\) is the profile of players’ network proposals, then the \(m\)-tuple \(G_D := (G_d)_{d \in D}\) is such that for \(d \in S\), \(G_d\) may differ substantially from \(\tilde{G}\), but for \(d \notin S\), \(G_d = \tilde{G}\).

Before we give a formal statement of the primitives of our game theoretic model of network formation, consider the following example of a state space for dynamic games of network formation over risky networks.

**Example 1** (A Risky Network) Assume that the set of arc types, \(A\), is a closed, convex subset of \(\mathbb{R}^3\) and that the realized connection, \(\tilde{a} \in A\), between nodes \(i\) and \(j\) is given by

\[ \tilde{a} = \alpha_{ij}(a, \tilde{\epsilon}_{ij}), \]

where the arc function

\[ \alpha_{ij}(\cdot, \cdot) : A(ij) \times R \rightarrow A(ij) \]

is continuous in feasible arc types \(a\) and measurable in disturbances, \(\tilde{\epsilon}_{ij}\), and where the real-valued disturbances, \(\tilde{\epsilon}_{ij}\), are distributed according to a continuous, conditional density

\[ 9\text{See Hildenbrand (1974, p. 45) and Sub-section 4.2.1.} \]

16
$h_{ij}(\cdot|G,S)$ with expected value

$$E_{(G,S)}(\varepsilon_{ij}) := \int_{-\infty}^{\infty} \varepsilon h_{ij}(\varepsilon|G,S)d\varepsilon \text{ for each } (G,S) \in G \times \mathcal{F}.$$ 

If we also assume that for all pre-connections, $(i,j) \in N \times N$, the arc function, $\alpha_{ij}(\cdot,\cdot)$ is such that for all network-coalition pair, $(G,S) \in G \times \mathcal{F}$,

$$\int_{-\infty}^{\infty} \alpha_{ij}(a,\varepsilon) h_{ij}(\varepsilon|G,S)d\varepsilon = a \text{ for all } a \in A(ij)$$

then we say that the arc connection function, $\alpha_{ij}(a,\cdot)$, is unbiased - meaning that we expect that the intended connection to survive implementation riskiness. We will assume that the disturbances, $\varepsilon_{ij}$, are independent for distinct pre-connections (i.e., for all $(i,j) \neq (i',j')$, $\varepsilon_{ij}$ and $\varepsilon_{i'j'}$ are independent). In this case it is easy to show that for each network and coalition pair, $(G,S) \in G \times \mathcal{F}$, the collection of random variables, $\{\alpha_{ij}(a,\cdot) : a \in G(ij)\}$, together with the collection of continuous densities, $\{h_{ij}(\cdot|G,S) : (i,j) \in \mathcal{D}(G)\}$, induce a continuous, conditional density, $h(\cdot|G,S)$ on the set of domain equivalent risky networks, $\mathcal{G}_{\mathcal{D}(G)}$, absolutely continuous with respect to a nonatomic probability measure on the risky networks, $\mathcal{G}_{\mathcal{D}(G)}$. 

17
4 Discounted Stochastic Games of Risky Network Formation

4.1 Primitives and Assumptions

An $m$-player, non-zero sum discounted stochastic game of network formation over $G$ with risky connections is defined by the following primitives:

\[
\{(\Omega, B(\Omega), (\lambda \times \mu), (\mathcal{P}(G), \mathcal{F}, \mathcal{P}(\Phi_d(\cdot), z_d(\cdot, \cdot), \beta_d)_{d\in D}, q(\cdot, \cdot, \cdot))\},
\]

where

(1) $D$ is a finite set of players consisting of $|D| = m$ players and $\beta_d \in (0, 1)$ is player $d$’s discount rate, $G \subset P_f(K)$ is the compact metric space of unique connections networks with Hausdorff metric $h$, $F \subset 2^D$ is the feasible set of coalitions;

(2) $(\Omega, B(\Omega), (\lambda \times \mu))$ is the probability space of states where $\Omega$ is a product space given by

\[
\Omega := G \times (G \times F),
\]

a compact metric space with metric $d_\Omega := h + h_F$ and typical element

\[
\omega = (\tilde{G}, (G, S)),
\]

with risky state, $\tilde{G}$, contained in the domain equivalence class, $G_{D(G)}$, determined by $G$, where $B(\Omega)$ is the product $\sigma$-field,

\[
B(\Omega) = B(G) \times B(G) \times P(F)
\]

with Borel $\sigma$-field $B(G)$, generated by the metric $h$ and discrete $\sigma$-field, $P(F)$, of all subsets of $F$, and where

\[
\lambda \times \mu := \lambda \times \nu \times \eta
\]

is a product probability measure on $\Omega$ with $\lambda \in \mathcal{P}(G)$, the probability measure governing risky states, being nonatomic, and $\mu = \nu \times \eta \in \mathcal{P}(G \times F)$, the product probability measure governing the regular states, being such that $\eta \in \mathcal{P}(F)$ assigns positive probability to all coalitions (i.e., $\eta(S) > 0$ for all $S \in F$);

(3) $G$ is the space of all potential network proposals available to player $d$ with typical element $G_d$;

(4) $\Phi_d(\cdot)$ is the feasible proposal correspondence, a measurable set-valued mapping from the state space $\Omega$ into the nonempty, $h$-closed subsets of $G$ with graph

\[
Gr\Phi_d(\cdot) := \{(\omega, G_d) \in \Omega \times G : G_d \in \Phi_d(\omega)\},
\]
such that if \( \omega := (\tilde{G}, (G, S)) \), then for \( d \in S \), \( \tilde{G} \in \Phi_d(\omega) = \Phi_d(\tilde{G}, (G, S)) \), and for \( d \notin S \), \( \{G\} = \Phi_d(\omega) = \Phi_d(G, (G, S)) \).

Thus, \( \Phi_d(\omega) \subseteq \mathbb{G} \) is the compact subset of feasible network proposals available to player \( d \) in each state \( \omega \in \Omega \).

Because \( \Phi_d(\cdot) \) is compact-valued and maps from a separable metric space \( \Omega \) to a compact metric space \( \mathbb{G} \), the measurability of \( \Phi_d(\cdot) \) is equivalent to \( \Phi_d(\cdot) \) having a measurable graph (e.g., Lemma 1.7 in Nowak 1984). Thus, the measurability of \( \Phi_d(\cdot) \) is equivalent to \( \text{Gr} \Phi_d(\cdot) \in \mathcal{B}(\Omega) \times \mathcal{B}(\mathbb{G}) \).

Letting

\[
\mathbb{G}^m := \mathbb{G} \times \cdots \times \mathbb{G},
\]

\( m \)-times

\( \mathbb{G}^m \) is the compact subset of all possible proposal profiles available to players, with typical element \( G_D = (G_d, G_{-d}) \in \mathbb{G}^m \). Letting

\[
\Phi(\cdot) := \Phi_1(\cdot) \times \cdots \times \Phi_m(\cdot) := \prod_{d \in D} \Phi_d(\cdot),
\]

\( \Phi(\omega) \subseteq \mathbb{G}^m \) is the compact subset of feasible proposal profiles \((m\text{-tuples})\) available to players in state \( \omega \). Because \( \Phi(\cdot) \) is a measurable set-valued mapping (Lemma 18.4, Aliprantis-Border, 2006) from the state space \( \Omega \) into the nonempty, compact subsets of \( \mathbb{G}^m \), the graph of \( \Phi(\cdot) \), given

\[
\text{Gr} \Phi(\cdot) := \{ (\omega, G_D) \in \Omega \times \mathbb{G}^m : G_D \in \Phi(\omega) \},
\]

is contained in the Borel product \( \sigma \)-field \( \mathcal{B}(\Omega) \times \mathcal{B}(\mathbb{G}^m) = \mathcal{B}(\Omega) \times \underbrace{\mathcal{B}(\mathbb{G}) \times \cdots \times \mathcal{B}(\mathbb{G})}_{\text{m-times}} \).

(5) \( r_d(\cdot, \cdot) \) is player \( d \)'s real-valued immediate payoff function defined on \( \Omega \times \mathbb{G}^m \), such that for each player \( d \in D \) (i) \( |r_d(\omega, G_D)| \leq M \) for all \( (\omega, G_D) \in \Omega \times \mathbb{G}^m \), (ii) \( r_d(\cdot, G_D) \) is measurable on \( \Omega \) for all \( G_D \in \mathbb{G}^m \), and (iii) \( r_d(\omega, \cdot) \) is continuous on \( \mathbb{G}^m \) for all \( \omega \in \Omega \);

(6) \( q(\cdot, \cdot) \) is the law of motion,

\[
((\tilde{G}, (G, S)), G_D) \rightarrow q(\cdot | (\tilde{G}, (G, S)), G_D)
\]

and is given by

\[
q(d(\tilde{G}', (G', S')) | (\tilde{G}, (G, S)), G_D) := \varepsilon(d\tilde{G}' | G', S') \rho(d(G', S') | (\tilde{G}, (G, S)), G_D)
\]

or

\[
q(d\omega' | \omega, G_D) := \varepsilon(d\tilde{G}' | G', S') \rho(d(G', S') | \omega, G_D),
\]

\(^{10}\)We say that \( \Phi_d(\cdot) \) is measurable if

\[
\Phi_d^{-1}(E) := \{ \omega \in \Omega : \Phi_d(\omega) \cap E \neq \emptyset \} \in \mathcal{B}(\Omega)
\]

for \( E \subseteq \mathbb{G} \) open (sometimes called weak or lower measurability). Because \( \mathbb{G} \) is compact, we also know that \( \Phi_d(\cdot) \) is measurable if and only if \( \Phi_d^{-1}(F) \in \mathcal{B}(\Omega) \) for \( F \subset \mathbb{G} \) closed (see Nowak 1984, p. 16, or Aliprantis and Border 2006, p. 595).
where \( \omega = (\bar{G}, (G, S)) \) denotes the current state and \( \omega' = (\bar{G}', (G', S')) \) denotes the coming state. Thus, depending on the regular state \((G', S')\) chosen by the regular probability measure, \( \rho(d(G', S')|\omega, G_D) \), in current state \( \omega := (\bar{G}, (G, S)) \) given network proposal profile \( G_D \in \Phi(\omega) = \Phi(\bar{G}, (G, S)) \), the choice of the coming risky state \( \bar{G}' \) will be governed by the probability measure, \( \varepsilon(d\bar{G}'|G', S') \). We will assume that

1. \( (\varepsilon(\cdot|G', S')) << \lambda \) for all \((G', S') \in \mathcal{G} \times \mathcal{F})\) for all regular states \((G', S') \in \mathcal{G} \times \mathcal{F})\) the probability measure, \( \varepsilon(dG'|G', S') \), governing risky states, is absolutely continuous with respect to the nonatomic probability measure \( \lambda \in \mathcal{P}(\mathcal{G}) \), and for all sets \( E \in \mathcal{B}(\mathcal{G}) \times \mathcal{P}(\mathcal{F}) \), \( \varepsilon(E, \cdot) \) is measurable on \( \mathcal{G} \times \mathcal{F} \),

2. \( (\rho(\cdot|\omega, G_D) << \nu \times \eta) \) for all \((\omega, G_D) \in \Omega \times \mathcal{G}^m)\) for all state-network proposal profiles, \((\omega, G_D) \in \Omega \times \mathcal{G}^m)\) the probability measure, \( \rho(d(G', S')|\omega, G_D) \), governing regular states, is absolutely continuous with respect to the product probability measure \( \mu := \nu \times \eta \in \mathcal{P}(\mathcal{G} \times \mathcal{F}) \), and for all sets \( E \in \mathcal{B}(\mathcal{G}) \times \mathcal{P}(\mathcal{F}) \), \( \rho(E, \cdot) \) is measurable on \( \Omega \times \mathcal{G}^m \), and

3. \((L_1 \text{ continuity})\) the collection of probability density functions of \( q(\cdot|\omega, G_D) \) with respect to \( \lambda \times \mu := \lambda \times \nu \times \eta \), given by

\[
H_{(\lambda \times \mu)} := \{h(\cdot|\omega, G_D) : (\omega, G_D) \in \Omega \times \mathcal{G}^m\}
\]

is such that for each state \( \omega \in \Omega \) and a.e. \((\lambda \times \mu)\) in \( \omega' = (\bar{G}', (G', S'))\) the function \( G_D \rightarrow h(\omega'|\omega, G_D) \) is continuous.

We will refer to our list of assumptions above concerning discounted stochastic games of network formation as \([A-2]\), and we will refer to them individually as \([A-2](j)\) for \( j = 1, 2, \ldots, 6 \).

### 4.2 Discussion of Primitives and Assumptions

#### 4.2.1 The Existence of a Nonatomic Probability Measure on \( \mathcal{G} \)

We have the following unique decomposition of the space of networks, \( \mathcal{G} \), into disjoint domain equivalence classes,

\[
\mathcal{G} = \bigcup_{C \in \mathcal{P}(N \times N)} \mathcal{G}_C
\]

where

\[
\mathcal{G}_C := \{G \in \mathcal{G} : \mathcal{D}(G) = C\}.
\]

Because \( \mathcal{G}_C \) is an uncountable and compact metric space, we can equip \( \mathcal{G}_C \) with a nonatomic probability measure, \( \lambda_C(\cdot) \), and therefore, we can equip \( \mathcal{G}_C \) with a probability measure, \( \lambda_C(\cdot) \), such that \( \lambda_C(G) = 0 \) for all \( G \in \mathcal{G}_C \). Because \( \mathcal{G} = \bigcup_{C \in \mathcal{P}(N \times N)} \mathcal{G}_C \) where each \( \mathcal{G}_C \) is a nonatomic probability space, if we equip \( \mathcal{G} \) with the probability measure \( \lambda(\cdot) \) given by

\[
\lambda(\mathbb{E}) := \sum_{C \in \mathcal{P}(N \times N)} \lambda_C(\mathbb{E} \cap \mathcal{G}_C) / |\mathcal{P}(N \times N)| \text{ for all } \mathbb{E} \in \mathcal{B}(\mathcal{G}_C),
\]

\( \mathbb{E} \subset \mathcal{G}_C \) is an atom of \( \mathcal{G}_C \) relative to \( \lambda_C(\cdot) \) if the following implication holds: if \( \lambda_C(\mathbb{E}) > 0 \), then \( \mathbb{E} \subset H \) implies that \( \lambda_C(H) = 0 \) or \( \lambda_C(\mathbb{E} - H) = 0 \). If \( \mathcal{G}_C \) contains no atoms relative to \( \lambda_C(\cdot) \), \( \mathcal{G}_C \) is said to be atomless or nonatomic. Because each domain equivalence class, \( \mathcal{G}_C \), is a compact metric space - and hence separable - \( \lambda_C(\cdot) \) is atomless (or nonatomic) if and only if \( \lambda_C(\bigcup_{\mathcal{G}_C} G) = 0 \) for all \( G \in \mathcal{G}_C \) (see Hildenbrand, 1974, pp 44-45).
for $\mathcal{B}(G_C)$ the Borel $\sigma$-field in $G_C$ generated by the Hausdorff metric $h$, then $(G, \mathcal{B}(G), \lambda(\cdot))$ is a nonatomic probability space of unique connections networks. Thus, contained in the space of all probability measures, $\mathcal{P}(G)$, on the compact metric space of unique connections networks, $(G, h)$, is at least one that is nonatomic.

### 4.2.2 The Regular State Space

We have taken as the regular state space the set $G \times F$ of all feasible network-coalition pairs. If the current or status quo state is $\omega := (\tilde{G}, (G, S))$ with risky state $\tilde{G}$ and regular state $(G, S)$ then $G$ is the network whose choice was governed by the regular law of motion (i.e., the law of motion without noise), $\rho(\cdot, \cdot, \cdot)$, and $\tilde{G}$ is the network, in current state $\omega$, that is arrived at from $G$ via the resolution of the risky state - a resolution governed by the probability measure, $\varepsilon(\cdot | G, S)$. It is this final noise-determined network, $G$, that the members of the current coalition $S$ - the players whose turn it is to move - take as their starting point when putting forth substantive network proposals, $G_S$. But coalition $S$’s influence on the process of network formation is limited to choosing, via its profile of proposals, $G_S$, the regular-state Markov transition,

$$G_S \rightarrow \rho(\cdot | (\tilde{G}, (\cdot)), G_S, G_{-S}),$$

which, together with the regular current state $(G, S)$, govern the choice of the coming (or next) regular state $(G', S')$ - via the probability measure governing regular states,

$$\rho(d(G', S') | (\tilde{G}, (G, S)), G_S, G_{-S}) := \rho(d(G', S') | \omega, G_D),$$

and it is through the next regular state, $(G', S')$, that coalition $S$ indirectly influences nature’s choice of the next risky state, $\tilde{G}'$, via nature’s risky-state probability measure $\varepsilon(dG' | G, S')$. As we discussed above, $\rho(d(G', S') | \omega, G_D)$ represents the bargaining risk, while $\varepsilon(dG' | G, S')$ represents the implementation risk.

The regular state space $G \times F$ is a compact metric space under the metric $h + d_F$. Letting $\mathcal{B}(G \times F) = \mathcal{B}(G) \times P(F)$ be the Borel product $\sigma$-field generated by the metric $h + d_F$, we have equipped the regular state measurable space

$$(G \times F, \mathcal{B}(G) \times P(F))$$

with the product probability measure

$$\mu = \nu \times \eta$$

where $\nu$ is a probability measure on networks $G$ and where $\eta$ is a probability measure on feasible coalitions is such that $\eta(S) > 0$ for all $S \in F$. Thus, we have as our regular state space, the probability space

$$(G \times F, \mathcal{B}(G) \times P(F), \nu \times \eta),$$

a compact metric space with metric $h + d_F$ and typical element $(G, S)$.

### 4.2.3 The Stochastic Continuity Assumptions

The $L_1$ continuity of the collection of density functions,

$$H_{(\lambda \times \mu)} := \{ h(\cdot | \omega, G_D) : (\omega, G_D) \in \Omega \times A \},$$
in $G_D$ implies via Scheffé’s Theorem (see Billingsley, 1986, Theorem 16.11) that
\[
\sup_{E \in \mathcal{B}(\Omega)} |q(E|\omega, G_D^n) - q(E|\omega, G_D^m)|_R
\leq \int_{\Omega} |h(\omega'|\omega, G_D^n) - h(\omega'|\omega, G_D^m)|_R d(\lambda \times \mu)(\omega') \to 0,
\]
for any sequence of network profiles $\{G_D^n\}_n$ in $\Phi(\omega)$ converging to $G_D \in \Phi(\omega)$. Thus, $G_D^n \to G_D$ implies that
\[
\sup_{E \in \mathcal{B}(\Omega)} |q(E|\omega, G_D^n) - q(E|\omega, G_D^m)|_R \to 0,
\]
sometimes written $\|q(\cdot|\omega, G_D^n) - q(\cdot|\omega, G_D^m)\|_\infty \to 0$ (also, see Lasserre, 1998).

### 4.2.4 The Space of Action Profiles and the Space of Payoff Profiles

Let $h_G := h + \cdots + h$ be the metric on space of network profiles, $\mathbb{G}^m$. We will denote by $\to_h$ sequential convergence in $\mathbb{G}^m$ with respect to the metric $h_G$. Also, let $d_X := \sum_{d \in D} d_{X_d}$ be the metric on space of payoff profiles, $X := \prod_{d \in D} X_d$, where each metric $d_{X_d}$ is given by $d_{X_d}(x, x') := |x - x'|_R$ (i.e., the absolute value of payoff differences). Finally, let $d_{w^* \times \mathbb{G}^m} := d_{w^*} + h_G$ and $d_{w^* \times X} := d_{w^*} + d_X$. Thus, the spaces, $(\mathcal{L}_X^\infty \times A, d_{w^* \times A})$ and $(\mathcal{L}_X^\infty \times X, d_{w^* \times X})$, are compact metric spaces.

### 4.3 The Space of Value Function Profiles

Let $v = (v_d)_{d \in D}$ denote the $m$-tuple of player value functions, where each player’s set of value functions is given by $\mathcal{L}_{X_d}$, the set of all $(\lambda \times \mu)$-measurable classes of $\mathcal{B}(\Omega)$-measurable functions, $v(d) : \Omega \to X_d$, where $X_d := [-M, M]^m$. Because the Borel $\sigma$-field $\mathcal{B}(\Omega)$ is countably generated, the space of $(\lambda \times \mu)$-equivalence classes of $(\lambda \times \mu)$-integrable functions, $\mathcal{L}_{X_d}$, is separable. As a consequence, the set of value function $(\lambda \times \mu)$-equivalence classes $\mathcal{L}_{X_d}$ is a compact, convex, and metrizable subset of $\mathcal{L}_R^\infty$ for the weak star topology, denoted by $w^*_d$. Let $d_{w^*}$ be a metric compatible with the relative $w^*$-topology on $\mathcal{L}_{X_d}^\infty$. Thus, $(\mathcal{L}_{X_d}^\infty, d_{w^*})$ is a compact metric space. Next let
\[
\mathcal{L}_{X}^\infty := \prod_{d \in D} \mathcal{L}_{X_d}^\infty,
\]
where $X := \prod_{d \in D} X_d := [-M, M]^m$ and equip $\mathcal{L}_X^\infty$ with the metric $d_{w^*} := \sum_{d \in D} d_{w^*_d}$. Note that the sum metric, $d_{w^*} := \sum_{d \in D} d_{w^*_d}$, is compatible with the relative weak star product topology on $\mathcal{L}_X^\infty$. Thus, $(\mathcal{L}_X^\infty, d_{w^*})$, the space of value function profiles, is also a compact metric space. We will denote by $\to_{w^*}$ or $\to$ sequential $w^*$-convergence in $\mathcal{L}_X^\infty$.

### 4.4 Random Networks and Markov Strategies

We will refer to any probability measure, $\sigma(\cdot) \in \mathcal{P}(\mathbb{G})$, with support contained in the space of networks, $\mathbb{G}$, as a random network. A Markov strategy for player $d \in D$, $\omega \to_{d_d}(\cdot|\omega)$, is a measurable function defined on $\Omega$ such that in each state $\omega \in \Omega$, $\sigma_d(\cdot|\omega)$, is a probability measure with support contained in the feasible subset of networks, $\Phi_d(\omega) \subset \mathbb{G}$ - a subset

\[\text{(31)}\]

\[\text{Each player’s value function } v_d \in \mathcal{L}_{X_d} \text{ is the key to the player’s ability to assign a value to following a particular strategy over an infinite time horizon. Part of our mission here is to show that there exists an equilibrium profile of value functions - as well as an equilibrium profile of stationary strategies.}\]
available to player $d$ in state $\omega$. Thus, under Markov strategy $\sigma_d(\cdot|\cdot)$ in any state $\omega \in \Omega$, player $d$ chooses feasible random network $\sigma_d(\cdot|\omega) \in \mathcal{P}(\Phi_d(\omega))$, with $\sigma_d(\Phi_d(\omega)|\omega) = 1$. We will denote by

$$F_d := \Sigma(\Omega, \mathcal{P}(\Phi_d(\cdot)))$$

(32)

the set of all Markov strategies for player $d$, and we will denote by

$$F := \prod_d F_d$$

(33)

the set of all Markov strategy profiles.

The feasible random networks correspondence

$$\omega \rightarrow \mathcal{P}(\Phi_d(\omega))$$

is a set-valued mapping from the state space $\Omega$ into the nonempty, compact, convex subsets of $\mathcal{P}(\mathcal{G})$ with measurable graph, $Gr\mathcal{P}(\Phi_d(\cdot)) \subseteq \mathcal{B}(\Omega) \times \mathcal{B}(\mathcal{P}(\mathcal{G}))$. In state $\omega$, $\Phi_d(\omega) \subseteq \mathcal{G}$ is player $d$’s feasible set of networks and $\mathcal{P}(\Phi_d(\omega))$ is player $d$’s corresponding feasible set of random networks. Letting $\mathcal{M}_n^w(\mathcal{G})$ denote the set of all bounded signed measures on $(\mathcal{G}, \mathcal{B}(\mathcal{G}))$ with the narrow topology (i.e., the $w^*$-topology), $\mathcal{M}_n^w(\mathcal{G})$ is the separable norm dual of the separable Banach space of continuous functions defined on the compact metric space $\mathcal{G}$ (see III.54, Dellacherie and Myer 1975). In $\mathcal{M}_n^w(\mathcal{G})$, the subset of random networks, $\mathcal{P}(\mathcal{G})$, is a convex and $w^*$-compact subset, metrizable for the relative narrow topology (see III.60, Dellacherie and Myer 1975). Moreover, by Theorem 3 in Himemlberg and Van Vleck (1975), the feasible random networks correspondence,

$$\omega \rightarrow \mathcal{P}(\Phi_d(\omega))$$

has nonempty, compact, convex values (with respect to the metrizable narrow topology) and is measurable (upper hemicontinuous) if and only if the underlying feasible networks correspondence $\Phi_d(\cdot)$ is measurable (upper hemicontinuous).

### 4.5 Expected Payoffs Under Stationary Markov Strategies

Under Markov strategy profile $\sigma(\cdot|\cdot)$, player $d$’s immediate payoff in state $\omega \in \Omega$ is

$$r_d(\omega, (\sigma_1(\cdot|\omega), \ldots, \sigma_m(\cdot|\omega)) := \int_{G^m} r_d(\omega, (G_1, \ldots, G_m)) \sigma_1(dG_1|\omega) \times \cdots \times \sigma_m(dG_m|\omega).$$

$$= \int_{G^m} r_d(\omega, G_D) \sigma(dG_D|\omega)$$

$$= r_d(\omega, \sigma(\omega)).$$

(34)

If under Markov strategy profile, $\sigma(\cdot)$, players choose random network profile $\sigma(\omega)$ in state $\omega$, then the law of motion (i.e., nature) $q(\cdot|\cdot, \cdot)$ chooses the next state $\omega'$ according to the probability measure

$$q(E|\omega, \sigma(\omega)) = \int q(E|\omega, G_D) \sigma(dG_D|\omega)$$

$$= \int q(E|\omega, (G_1, \ldots, G_m)) \sigma_1(dG_1|\omega) \times \cdots \times \sigma_1(dG_m|\omega)$$

$$= \int_E \left[ \int_{G^m} h(\omega'|\omega, (G_1, \ldots, G_m)) \sigma_1(dG_1|\omega) \times \cdots \times \sigma_1(dG_m|\omega) \right] d(\lambda \times \mu)(\omega').$$

Let

$$r_d^n(\sigma)(\omega) := \begin{cases} r_d(\omega, \sigma(\omega)) & \text{for } n = 1 \\ \int_\Omega r_d(\omega', \sigma(\omega')) q^{n-1}(\omega'|\omega, \sigma(\omega)) & \text{for } n \geq 2, \end{cases}$$

(36)
denote the \( n \)th period expected payoff to player \( d \) under Markov strategy profile \( \sigma(\cdot) \) starting at state \( \omega \) given law of motion \( q(\cdot|\cdot, \cdot) \). Here, for \( n \geq 2 \), \( q^n(\cdot|\cdot, \cdot) \) is defined recursively by
\[
q^n(E|\omega, \sigma(\omega)) = \int_{\Omega} q(E|\omega', \sigma(\omega))q^{n-1}(\omega'|\omega, \sigma(\omega)).
\]
The discounted expected payoff to player \( d \), with discount rate \( \beta_d \in [0, 1) \), over an infinite time horizon under Markov strategy profile \( f(\cdot) \) starting at state \( \omega \) is given by
\[
E_d(\sigma)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1}(\omega)(\sigma(\omega)).
\]

5 Stationary Markov Equilibria and the Underlying One-shot Game

A discounted stochastic game of network formation over stationary Markov strategies is given by
\[
NDSG := (\Omega, E_d(\cdot)(\cdot), F_d)_{d \in D}.
\]

**Definition 3 (Stationary Markov Equilibria)**

A Markov strategy profile \( \sigma(\cdot) = (\sigma_d^+, \sigma_d^-) \in F \) is a stationary equilibrium of the discounted stochastic game of network formation, \( NDSG \), if no player \( d \) can unilaterally benefit from changing his Markov strategy \( \sigma_d^+ \) to any other Markov strategy. This will be the case provided that for all players \( d \) and in all initial states \( \omega \),
\[
E_d(\sigma_d^+, \sigma_d^-)(\omega) \geq E_d(\sigma_d, \sigma_d^-)(\omega) \text{ for all } \sigma_d \in F_d.
\]
The starting point for our search for a proof of existence of a stationary Markov equilibrium is the state-contingent, one-shot game given by
\[
G(\omega, v) := (P(\Phi_d(\omega)), u_d(\omega, \cdot)(v_d))_{d \in D}
\]
where for each state \( \omega \in \Omega \), player \( d \)'s strategy set is \( P(\Phi_d(\omega)) \) and player \( d \)'s payoff function is
\[
u_d(\omega, \sigma)(v_d) := (1 - \beta_d)r_d(\omega, (\sigma_d, \sigma_d^-)) + \beta_d \int_{\Omega} v_d(\omega')q(\omega'|\omega, (\sigma_d, \sigma_d^-)).
\]

It follows from Blackwell (1965) that in order to prove the existence of a stationary Markov equilibrium, \( \sigma(\cdot) = (\sigma_d^+, \sigma_d^-) \in F \), for the discounted stochastic game of network formation, \( NDSG \), it is necessary and sufficient to prove that there exists an \( m \)-tuple of value functions,
\[
v^*(\cdot) = (v_d^*(\cdot), v_d^-) \in L_X^N,
\]
(i.e., a value function profile) such that together with \( m \)-tuple of Markov strategies, \( \sigma^*(\cdot) = (\sigma^*_N(\cdot), \sigma^*_D(\cdot)) \), (i.e., a Markov strategy profile), the pair \( (\sigma^*, v^*) \) satisfies the following equations for all players simultaneously. Consider \( L \) and (b) we know by Blackwell that

Thus, in game \( G \) and because

Thus, \( B \) has an incentive to unilaterally deviate from strategy \( \sigma^*_d(\cdot) \) to any other Markov strategy \( \sigma_d(\cdot) \), or to any other strategy, Markov or otherwise (see Blackwell, 1965, Theorem 6) - and because \( (\sigma^*, v^*) \) satisfies (a) and (b) we know by Blackwell that \( G_{(\omega, v^*)} \) is the correct game to play.

5.1 The Continuity of One-Shot Payoffs

We begin with the following Lemma concerning continuity of one-shot payoff functions.

**Lemma 1 (The Continuity Lemma)**

Suppose assumptions [A-1] and [A-2] hold and let \( \{v^n, G^n_D\}_{n \in \mathbb{N}} \) be any sequence in \( \mathcal{X} \times \mathcal{G}^m \). If \( v^n \to v^* \) and \( G^n_D \to G^*_D \), then for each state \( \omega \in \Omega \) and each player \( d \in D \)

\[
u_d(\omega, G^n_D)(v^n_d) \to \nu_d(\omega, G^*_D)(v^*_d).
\]

**Proof.** Let \( \{(v^n, G^n_D)\}_{n \in \mathbb{N}} \) be a sequence such that \( v^n \to v^* \) and \( G^n_D \to G^*_D \). Let \( \omega \) be given and fixed, and observe that for each players \( d \):

\[
\frac{|\nu_d(\omega, G^n_D)(v^n_d) - \nu_d(\omega, G^*_D)(v^*_d)|}{A^n} + \frac{|\nu_d(\omega, G^*_D)(v^*_d) - \nu_d(\omega, G^n_D)(v^n_d)|}{B^n}.
\]

We will carry out our proof for one player \( d \), keeping in mind that the argument holds for all players simultaneously. Consider \( B^n \) first. We have

\[
B^n = \beta_d \int_{\Omega} v^n_d(\omega')q(\omega'|\omega, G^n_D) - \int_{\Omega} v^*_d(\omega')q(\omega'|\omega, G^*_D).\]

Let \( h(\cdot, G^*_D) \) be a density of \( q(\cdot, G^*_D) \) with respect to \( (\lambda \times \mu) \). Given that \( v^n_d \to v^*_d \), we have

\[
\int_{\Omega} v^n_d(\omega')q(\omega'|\omega, G^*_D) = \int_{\Omega} v^*_d(\omega')h(\omega'|\omega, G^*_D)d(\lambda \times \mu)(\omega')
\]


Thus, \( B^n \to 0 \). Next, consider \( A^n \). We have

\[
A^n \leq (1 - \beta_d) \frac{|\nu_d(\omega, G^n_D) - \nu_d(\omega, G^*_D)|}{A^n} + \beta_d \frac{|\nu_d(\omega', G^n_D) - \nu_d(\omega', G^*_D)|}{A^n}.
\]

25
Continuity of $r_d(\omega, \cdot)$ and $a^n \to a^*$ imply that $A_1^n \nrightarrow 0$. To see that $A_2^n \nrightarrow 0$, observe that
\[
\left| \int_{\Omega} v^n_d(\omega') q(\omega' | \omega, G^n_D) - \int_{\Omega} v^n_d(\omega') q(\omega' | \omega, G^n_D) \right| \\
\leq M \| q(\cdot | \omega, G^n_D) - q(\cdot | \omega, G^n_D) \|_\infty \nrightarrow 0.
\]

Let
\[
U((\omega, v), G_D) := (u_d(\omega, G_D)(v_d))_{d \in D}.
\]

From the Continuity Lemma, we have that $U((\omega, v), \cdot)$ is continuous in $G_D$ on $G^m$, and we can easily establish that $U((\cdot, \cdot), G_D)$ is $\mathcal{B}(\Omega) \times \mathcal{B}_{w^*}$-measurable (i.e. is jointly measurable) in $(\omega, v)$ on $\Omega \times \mathcal{L}^\infty_X$.\textsuperscript{13}

\textsuperscript{13}Here, $\mathcal{B}_{w^*}$ is the Borel $\sigma$-field generated by the metric $d_{w^*}$ compatible with the relative $w^*$ product topology in the space of value function profiles, $\mathcal{L}^\infty_X$. 26
6 Approximability and Stationary Markov Equilibria in Discounted Stochastic Games of Risky Network Formation

Given discounted stochastic game of network formation $NDSG$ and a particular state $\omega = (\tilde{G}, (G, S)) \in \Omega := G \times (G \times \mathcal{F})$, let

$$G(\omega, L_X) := \{ G(\omega, v) : v \in L_X^\infty \}$$

be the collection of underlying one-shot games, parameterized by value function profiles. Thus, given state $\omega$, if $v \in L_X^\infty$ is the prevailing value function profile, then the underlying one-shot game is given by

$$G(\omega, v) := (\mathcal{P}(\Phi_d(\omega)), u_d(\omega, \cdot)(v_d))_{d \in D}.$$ 

Also let

$$\{ G(\omega, L_X^\infty), N(\omega, \cdot), \mathcal{P}(\omega, \cdot) \}$$

be the corresponding one-shot triple, consisting of the parameterized collection, $G(\omega, L_X^\infty)$, together with its $\omega$-Nash correspondence, $N(\omega, \cdot)$, and its induced $\omega$-Nash payoff correspondence, $\mathcal{P}(\omega, \cdot)$, given by

$$v \rightarrow \mathcal{P}(\omega, v) := \{ U \in \mathbb{R}^m : U = U((\omega, v), \sigma), \sigma \in N(\omega, v) \}.$$ 

Given our assumptions [A-2], for each player $d$ the function

$$\sigma_d \rightarrow \int_{\Omega} v_d(\omega') q(\omega' | \omega, \sigma_d, \sigma_{-d})$$

is continuous on $\mathcal{P}(\Phi_d(\omega))$ for all states $\omega \in \Omega$ and for all value functions $v(\cdot) \in L_X^\infty$. Moreover, because each player’s payoff function,

$$\sigma_d \rightarrow u_d(\omega, \sigma_d, \sigma_{-d})(v_d),$$

is continuous and affine on $\mathcal{P}(\Phi_d(\omega))$, and because the feasible action correspondences, $\mathcal{P}(\Phi_d(\omega))$, are compact and convex, the one-shot game $G(\omega, v)$ always has a Nash equilibrium

$$\sigma^* \in N(\omega, v) \subset \mathcal{P}(\Phi(\omega))$$

for each state $\omega \in \Omega$ and value function profile $v(\cdot) \in L_X^\infty$. In fact, it is easy to show that the set of Nash equilibria, $N(\omega, v)$, is nonempty and compact for all $(\omega, v) \in \Omega \times L_X^\infty$. It is also easy to show that the Nash correspondence, $(\omega, v) \rightarrow N(\omega, v)$, is $\mathcal{B}(\Omega) \times \mathcal{B}_{\infty}$-measurable (i.e., jointly measurable in $(\omega, v)$) and that for each $\omega$ that the $\omega$-Nash correspondence, $v \rightarrow N(\omega, v)$, is upper semicontinuous (usc) with nonempty, compact values.14

---

14Joint measurability and upper semicontinuity can be shown by considering the Nash problem

$$\max_{\sigma \in \mathcal{P}(\Phi(\omega))} V((\omega, v), \sigma)$$

where

$$V((\omega, v), \sigma) := \sum_d \left( u_d(\omega, \sigma_d, \sigma_{-d})(v_d) - \max_{\sigma \in \mathcal{P}(\Phi_d(\omega))} u_d(\omega, \sigma, \sigma_{-d})(v_d) \right).$$

Joint measurability and usc then follow from the Continuity Lemma, the Berge Maximum Theorem (see Aliprantis and Border, 2006, 17.31), and from the Measurable Maximum Theorem (see Aliprantis and Border, 2006, 18.19) and Theorem 6.1 in Himmelberg (1975).
Definition 4 (Approximable Stochastic Games)

We say that a discounted stochastic game of network formation, $\mathcal{NDSG}$, satisfying assumptions [A-1] and [A-2] with one-shot triple, $\{G_{(\omega,\mathcal{L}_K)}, N(\omega, \cdot), P(\omega, \cdot)\}$, is approximable if for each state $\omega \in \Omega$ the $\omega$-Nash payoff correspondence, $P(\omega, \cdot)$, is such that for any $\varepsilon > 0$ there exists a $w^\ast$-$\rho_X$-continuous function

$$U^\varepsilon(\cdot): \mathcal{L}_\infty^\varepsilon \rightarrow \mathcal{X}$$

such that

$$\text{Gr}U^\varepsilon(\cdot) \subset B_{d_{w^\ast \times X}}(\varepsilon, \text{Gr}P(\omega, \cdot)).$$

Here, $B_{d_{w^\ast \times X}}(\varepsilon, \text{Gr}P(\omega, \cdot))$ is an $\varepsilon$-enlargement of the graph of the $\omega$-Nash payoff correspondence given by

$$B_{d_{w^\ast \times X}}(\varepsilon, \text{Gr}P(\omega, \cdot)) := \{(v, x) \in \mathcal{L}_\infty^\varepsilon \times X : \text{dist}_{d_{w^\ast \times X}}((v, x), \text{Gr}P(\omega, \cdot)) < \varepsilon\},$$

where

$$\text{dist}_{d_{w^\ast \times X}}((v, x), \text{Gr}P(\omega, \cdot)) := \inf_{(v', x') \in \text{Gr}P(\omega, \cdot)} [\rho_{w^\ast}(v, v') + \rho_x(x, x')].$$

Thus if $\mathcal{NDSG}$ is approximable, then for every $(v, x) \in \text{Gr}U^\varepsilon(\cdot)$ there exists $(v', x') \in \text{Gr}P(\omega, \cdot)$ such that

$$d_{w^\ast \times X}((v, x), (v', x')) := d_{w^\ast}(v, v') + d_X(x, x') < \varepsilon.$$

Our main results are the following:

Theorem 2 (All Discounted Stochastic Games of Network Formation with Risky Connections Are Approximable)

All discounted stochastic game of network formation with risky connections satisfying assumptions [A-1] and [A-2] are approximable.

We then show that all approximable discounted stochastic games of network formation have stationary Markov equilibria.

Theorem 3 (All Approximable Discounted Stochastic Games of Network Formation Have Stationary Markov Equilibria)

Let $\mathcal{NDSG}$ be any approximable discounted stochastic game of network formation satisfying assumptions [A-1] and [A-2] with one-shot triple, $\{G_{(\omega,\mathcal{L}_K)}, N(\omega, \cdot), P(\omega, \cdot)\}$. Then $\mathcal{NDSG}$ has a stationary Markov equilibrium, $\sigma^\ast(\cdot) \in F$. Moreover, strategy profile $\sigma^\ast(\cdot) \in F$ is a stationary Markov equilibrium if and only if there exists a value function profile, $v^\ast(\cdot) \in \mathcal{L}_\infty$, such that for the pair $(\sigma^\ast(\cdot), v^\ast(\cdot))$,

$$\sigma^\ast(\omega) \in \mathcal{N}(\omega, v^\ast) \quad \text{and} \quad v^\ast(\omega) \in P(\omega, v^\ast), \quad (47)$$

where $\mathcal{N}(\omega, v^\ast) \subset P(\Phi(\omega))$ is the set of Nash equilibria and $P(\omega, v^\ast) \subset X$ is the set of induced Nash equilibrium payoffs for the one-shot game,

$$G_{(\omega,v^\ast)} := (P(\Phi_d(\omega)), u_d(\omega, \cdot)(v^\ast_d))_{d \in D}, \quad (48)$$

with player payoff functions given by

$$\begin{align*}
\sigma_d &\rightarrow u_d(\omega, \sigma_d, \sigma_{-d})(v^\ast_d) \\
&:= (1 - \beta_d)r_d(\omega, \sigma_d, \sigma_{-d}) + \beta_d \int \Omega v^\ast_d(\omega')q(\omega'|\omega, \sigma_d, \sigma_{-d}), \quad (49)
\end{align*}$$

for each player $d \in D$.
Our proof of existence rests upon our ability to carry out a second type of graphical approximation - a Caratheodory approximation. In particular, we must be able to graphically approximate the Nash payoff mapping, \((\omega, v) \rightarrow \mathcal{P}(\omega, v)\), with Caratheodory functions. A result due to Kucia and Nowak (2000) tells us that if the \(\omega\)-Nash payoff mapping, \(v \rightarrow \mathcal{P}(\omega, v)\), is approximable for each \(\omega\), then the Nash payoff mapping, \((\omega, v) \rightarrow \mathcal{P}(\omega, v)\), is Caratheodory approximable. We begin by defining what we mean by Caratheodory approximation. We then show that under assumptions [A-1] and [A-2], all approximable discounted stochastic games of network formation have stationary Markov equilibria. Finally, we show that all discounted stochastic games of network formation with risky connections satisfying assumptions [A-1] and [A-2] are approximable.

6.1 Caratheodory Approximation

We say that a function \(U(\cdot, \cdot) \equiv (U_d(\cdot, \cdot))_{d \in D}\),

\[
U(\cdot, \cdot) : \Omega \times \mathcal{L}_X^\infty \rightarrow X,
\]

is Caratheodory if it jointly measurable (i.e., \(\mathcal{B}(\Omega) \times \mathcal{B}_{w^*}\)-measurable) and for each state \(\omega\), \(U(\omega, \cdot)\) is \(w^*\)-continuous.

We say that a Caratheodory function \(U^\varepsilon(\cdot, \cdot)\) is a Caratheodory \(\varepsilon\)-approximation of the Nash payoff correspondence, \(\mathcal{P}(\cdot, \cdot) : \Omega \times \mathcal{L}_X^\infty \rightarrow \mathcal{P}(X)\) if for every \(\omega \in \Omega\)

\[
\text{Gr}U^\varepsilon(\omega, \cdot) \subset B_{w^* \times X}(\varepsilon) \setminus \text{Gr} \mathcal{P}(\omega, \cdot),
\]

meaning that for every \(\omega \in \Omega\) and every \(v \in \mathcal{L}_X^\infty\) there exists \(\overline{v} \in \mathcal{L}_X^\infty\) and \(\overline{U} \in \mathcal{P}(\omega, \overline{v})\) such that

\[
d_{w^*}(v, \overline{v}) + d_X(U^\varepsilon(\omega, v), \overline{U}) < \varepsilon.
\]

Now we have the results by Kucia and Nowak (2000) (i.e., the KN Lemma).

**Lemma 4** (KN Lemma on the Caratheodory Approximation of the Nash Payoff Correspondence)

Let \(\mathcal{NDSG}\) be a discounted stochastic game satisfying assumptions [A-1] with one-shot triple, \(\{\mathcal{G}(\omega, \mathcal{L}_X), \mathcal{N}(\omega, \cdot), \mathcal{P}(\omega, \cdot)\}\). If \(\mathcal{NDSG}\) is approximable, then the following statements are true:

1. The Nash payoff correspondence \((\omega, v) \rightarrow \mathcal{P}(\omega, v)\) is upper Caratheodory (i.e., jointly measurable in \((\omega, v)\) and \(w^*\)-upper semicontinuous in \(v\)).
2. The Nash payoff graph correspondence \(\omega \rightarrow \text{Gr} \mathcal{P}(\omega, \cdot)\) is measurable with nonempty closed values in \(\mathcal{L}_X^\infty \times X\).
3. For each \(\varepsilon > 0\) the Nash payoff correspondence, \(\mathcal{P}(\cdot, \cdot)\), has a Caratheodory \(\varepsilon\) approximation, \(U^\varepsilon(\cdot, \cdot)\).

Part (1) is straightforward. Part (2) of the Lemma is a restatement of Lemma 3.1(ii) in Kucia-Nowak (2000). Given that the \(\omega\)-Nash payoff correspondence, \(\mathcal{P}(\omega, \cdot)\), is approximable for all \(\omega\) (see Theorem 2 above), Part (3) of the Lemma is a restatement of Theorem 4.2 in Kucia-Nowak (2000).

Now we have the proof of our main result on the existence of stationary Markov equilibria of approximable discounted stochastic games of network formation.
6.2 The Existence of Stationary Markov Equilibria

By part (3) of the KN Lemma above, because \( \mathcal{P}(\cdot, \cdot) \) is upper Caratheodory and because \( \mathcal{P}(\omega, \cdot) \) is \( \varepsilon \)-approximable for each \( \omega \), \( \mathcal{P}(\cdot, \cdot) \) is Caratheodory \( \varepsilon \)-approximable for all \( \varepsilon > 0 \). Therefore, for each \( n \), let \( U^n(\cdot, \cdot) \) be a Caratheodory \( \frac{1}{n} \)-approximation of \( \mathcal{P}(\cdot, \cdot) \) and consider the sequence of functions,

\[
v \rightarrow T^n(v) := U^n(\cdot, v) \in \mathcal{L}_X^n.
\]

Observe that for each \( \omega \), \( T^n(\cdot) \) is a function from \( \mathcal{L}_X^n \) into \( \mathcal{L}_X^n \), and moreover, that for each \( n \) the function \( T^n(\cdot) \) is \( w^*-w^* \)-continuous (i.e., \( v^k \rightarrow v^* \) implies that \( T^n(v^k) \rightarrow T^n(v^*) \)). This is true because for each \( \omega \in \Omega \), \( U^n(\omega, v^k) \rightarrow U^n(\omega, v^*) \) in \( R^m \). By the Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 2006), for each \( n \), there exists a value function \( m \)-tuple \( v^n \in \mathcal{L}_X^n \) such that

\[
v^n = T^n(v^n) := U^n(\cdot, v^n).
\]

Thus, we have for each \( n \) a set, \( N^n \), of \( (\lambda \times \mu) \)-measure zero such that

\[
v^n(\omega) = U^n(\omega, v^n) \text{ for all } \omega \in \Omega \setminus N^n, \ (\lambda \times \mu)(N^n) = 0.
\]

Call the equation (52), one for each \( n \), the Caratheodory equation and call the sequence, \( \{v^n\} \), the Caratheodory fixed point sequence and let \( N^\infty := \bigcup_n N^n \) - so that, \( (\lambda \times \mu)(N^\infty) = 0 \).

For each Caratheodory approximating function, \( U^n(\cdot, \cdot) \), and fixed point, \( v^n(\cdot) \), pair consider the measurable function,

\[
\omega \rightarrow \min_{(v,U) \in Gr\mathcal{P}(\omega, \cdot)} [\rho_{w^*}(v^n, v) + \rho_X(U^n(\omega, v^n), U)].
\]

By part (2) of KN Lemma, the Nash payoff graph correspondence,

\[
\omega \rightarrow Gr\mathcal{P}(\omega, \cdot),
\]

is measurable, and therefore, by the continuity of the function

\[
(v,U) \rightarrow [\rho_{w^*}(v^n, \cdot) + \rho_X(U^n(\omega, v^n), \cdot)]
\]

on \( \mathcal{L}_X^n \times X \), there exists for each \( n \), a measurable selection of \( Gr\mathcal{P}(\omega, \cdot) \),

\[
\omega \rightarrow (\mathcal{P}_n^\omega, \mathcal{U}_n^\omega) \in \mathcal{L}_X^n \times X
\]

solving the minimization problem (53) state-by-state. Thus, for the measurable function, \( \omega \rightarrow (\mathcal{P}_n^\omega, \mathcal{U}_n^\omega) \), we have

\[
(\mathcal{P}_n^\omega, \mathcal{U}_n^\omega) \in Gr\mathcal{P}(\omega, \cdot) \text{ for all } \omega \in \Omega \}
\]

(i.e., \( \mathcal{U}_n^\omega \in \mathcal{P}(\omega, \mathcal{P}_n^\omega) \forall \omega \in \Omega \)),

and

\[
[\rho_{w^*}(v^n, \mathcal{P}_n^\omega) + \rho_X(U^n(\omega, v^n), \mathcal{U}_n^\omega)] = \min_{(v,U) \in Gr\mathcal{P}(\omega, \cdot)} [\rho_{w^*}(v^n, v) + \rho_X(U^n(\omega, v^n), U)],
\]

so that by part (3) of the KN Lemma,

\[
\rho_{w^*}(v^n, \mathcal{P}_n^\omega) + \rho_X(U^n(\omega, v^n), \mathcal{U}_n^\omega) < \frac{1}{n} \text{ for all } \omega \in \Omega.
\]
Therefore by (52) we have for each \( v^n \) in the fixed point sequence,

\[
\rho_w(v^n, \mathbf{w}^n) + \rho_X(v^\infty(\omega), \mathbf{U}^\infty_\omega) < \frac{1}{n}, \quad \text{for all } \omega \in \Omega \setminus N^\infty, \quad (\lambda \times \mu)(N^\infty) = 0,
\]

(56)

where the qualification, \( \omega \in \Omega \setminus N^\infty \), is from the Caratheodory equations (52) and where \( \mathbf{U}^\infty_\omega \in \mathcal{P}(\omega, \mathbf{u}^\infty_\omega) \) for all \( \omega \in \Omega \).

The proof will be complete if we can show that there exists a \( w^* \)-limit point, \( \mathbf{w}^* \in L^\infty_X \), of the fixed point sequence, \( \{v^n\}_n \), such that for some \( v^* \in L^\infty_X \) with \( v^*(\omega) = \mathbf{w}^*(\omega) \) a.e. \( [(\lambda \times \mu)] \), \( v^*(\omega) \in \mathcal{P}(\omega, v^*) \) for all \( \omega \in \Omega \). We have the following result.

**Theorem 5 (A Selection Theorem)**

Suppose assumptions [A-1] hold. Let \( \{v^n\}_n \) be a fixed point sequence generated by the sequence of Caratheodory approximating functions, \( \{U^n(\cdot, \cdot)\}_n \), for the Nash payoff correspondence, \( \mathcal{P}(\cdot, \cdot) \). Then for any

\[
\mathbf{w}^* \in L^\infty_X \{v^n\},
\]

there exists \( v^* \in L^\infty_X \) such that \( v^*(\omega) = \mathbf{w}^*(\omega) \) a.e. \( [(\lambda \times \mu)] \) and

\[
v^*(\omega) \in \mathcal{P}(\omega, v^*) \quad \text{for all } \omega \in \Omega.
\]

**Proof.** Let \( \{U^n(\cdot, \cdot), v^n(\cdot)\}_n \) be a sequence of Caratheodory approximating functions fixed point pairs and let

\[
\{\mathbf{w}^n(\cdot), \mathbf{v}^n(\cdot)\}^n
\]

be a corresponding sequence of optimal measurable selections solving state-by-state the problem,

\[
\min_{(v, U) \in \text{GrP}(\omega, \cdot)} \{\rho_w(v^n, v) + \rho_X(U^n(\omega, v^n), U)\}.
\]

By the \( w^* \)-compactness of \( L^\infty_X \), the set of \( w^* \)-limit point of the fixed point sequence, \( \{v^n\}_n, L^\infty_X \{v^n\} \) is nonempty. To simplify the notation we will assume that the fixed point sequence, \( \{v^n\}_n \subset L^\infty_X \), \( w^* \)-converges to some \( \mathbf{w}^* \). By the Caratheodory equations, we have for all \( n \),

\[
v^n(\omega) = U^n(\omega, v^n) \quad \text{for all } \omega \in \Omega \setminus N^\infty, \quad (\lambda \times \mu)(N^\infty) = 0,
\]

(57)

We also have for the fixed point sequence, \( \{v^n\}_n \),

\[
\left\{ \begin{array}{l}
\rho_w(v^n, \mathbf{w}^n) + \rho_X(U^n(\omega, v^n), \mathbf{U}^n_\omega) < \frac{1}{n}, \\
\end{array} \right.
\]

(58)

for all \( n \) and for all \( \omega \in \Omega \),

and therefore, by the Caratheodory equations,

\[
\left\{ \begin{array}{l}
\rho_w(v^n, \mathbf{w}^n) + \rho_X(v^n(\omega), \mathbf{U}^n_\omega) < \frac{1}{n}, \\
\end{array} \right.
\]

(59)

for all \( n \) and for all \( \omega \in \Omega \setminus N^\infty, \quad (\lambda \times \mu)(N^\infty) = 0 \).

Note that for each \( n \), \( \omega \to \mathbf{w}^n_\omega \) is \( L^\infty_X \)-valued, while \( \omega \to \mathbf{U}^n_\omega \) is \( R^m \)-valued. Note also that for all \( n \)

\[
\mathbf{U}^n_\omega \in \mathcal{P}(\omega, \mathbf{w}^n_\omega) \quad \text{for all } \omega \in \Omega.
\]

(60)
By Part A of (58), we have for all \(v^n \in \mathcal{L}^\infty_X\) in the fixed point sequence, \(\{v^n\}_n\), and for all \(\omega \in \Omega\),
\[
\rho_{v^n}(v^n, \bar{v}_n^\omega) < \frac{1}{n} \text{ for all } n.
\]
Thus as \(n \to \infty\), we have
\[
\bar{v}_n^\omega \xrightarrow{w} \bar{v}_n \text{ for all } \omega \in \Omega,
\]
and the same \(w^*\)-limit result holds for any further subsequence.

Next consider the measurable \(Ls\) correspondence,
\[
\omega \rightarrow Ls\{(\bar{v}_n^\omega, \bar{U}_n^\omega)\},
\]
for the sequence of optimal measurable selections, \(\{(\bar{v}_n^\omega, \bar{U}_n^\omega)\}_n\). Given (61),
\[
Ls\{(\bar{v}_n^\omega, \bar{U}_n^\omega)\} = \{\bar{v}_n^\omega\} \times Ls\{\bar{U}_n^\omega\} \text{ for all } \omega \in \Omega.
\]
By (60) and (61), because the \(\omega\)-Nash payoff correspondence, \(P(\omega, \cdot)\) is upper semicontinuous, we have for any selection, \(\omega \rightarrow \bar{U}_\omega\), of the \(Ls\) correspondence, \(\omega \rightarrow Ls\{\bar{U}_\omega\}\), that \(\bar{U}_\omega\) is a selection of the Nash payoff correspondence,
\[
\omega \rightarrow P(\omega, \bar{v}_n^\omega),
\]
that is,
\[
\bar{U}_\omega^\omega \in P(\omega, \bar{v}_n^\omega) \text{ for all } \omega \in \Omega.
\]
By Part C of expression (??), we have for all \(n\)
\[
\rho_X(v^n(\omega), \bar{U}_n^\omega) < \frac{1}{n} \text{ for all } \omega \in \Omega \setminus \mathcal{N}^\infty,
\]
where \(\bar{U}_n^\omega \in P(\omega, \bar{v}_n^\omega)\) for all \(n\) and for all \(\omega\). By Theorem 7.1 in Himmelberg (1971), for each \(n\), we have a measurable selection, \(\omega \rightarrow \bar{v}_n^\omega\), from the Nash mapping, \(\omega \rightarrow \mathcal{N}(\omega, \bar{v}_n^\omega)\), such that
\[
\bar{U}_n^\omega = \left( (1 - \beta_d)r_d(\omega, \bar{v}_n^\omega) + \beta_d \int_\Omega h(\omega'|\omega, \bar{v}_n^\omega)d(\lambda \times \mu)(\omega') \right)_{d \in D}.
\]
Let
\[
\omega' \rightarrow \bar{U}_n^\omega(\omega') := \left( (1 - \beta_d)r_d(\omega, \bar{v}_n^\omega) + \beta_d \bar{v}_n^\omega(\omega')h(\omega'|\omega, \bar{v}_n^\omega) \right)_{d \in D},
\]
where, given current state \(\omega\) and action profile \(\bar{v}_n^\omega\), \(h(\cdot|\omega, \bar{v}_n^\omega)\) is the density of the probability measure, \(q(\omega'|\omega, \bar{v}_n^\omega)\), for the coming state with respect to the dominating measure \((\lambda \times \mu)\) and note that for all \(n\) and for all \(\omega \in \Omega\),
\[
\bar{U}_n^\omega = \int_\Omega \bar{U}_n^\omega(\omega')d(\lambda \times \mu)(\omega').
\]
Now consider the sequence \(\{\bar{U}_n^\omega\}_n \subset \mathcal{L}^\infty_X\). By Komlos’ Theorem (1967) we can assume WLOG that \(\{\bar{U}_n^\omega\}_n\) \(K\)-converges. Thus, there exists a function, \(\hat{U}_n\), in \(\mathcal{L}^\infty_X\) and a set, \(\hat{\mathcal{N}}\), of measure zero such that for all \(\omega \in \Omega \setminus \hat{\mathcal{N}}\),
\[
\hat{U}_n^\omega := \frac{1}{n} \sum_{k=1}^n \bar{U}_n^\omega \xrightarrow{R_m} \hat{U}_\omega,
\]
with the same arithmetic mean limit results being true for any further subsequence - although possibly with a different set of measure zero.

32
Next, due to the $w^*$-compactness of $\mathcal{L}^\infty$, we can assume WLOG that $\overline{U}(\cdot) \overset{w^*}{\longrightarrow} U^*_*(\cdot) \in \mathcal{L}^\infty$. Thus, we have
\begin{align}
v^n(\cdot) \overset{w^*}{\longrightarrow} v^*(\cdot), \\
\overline{U}(\cdot) \overset{w}{\longrightarrow} U^*_*(\cdot),
\end{align}
and we know (for example, see Page, 1991) that for some set of measure zero, $N^*$, we have for all $\omega \in \Omega \setminus N^*$,
\begin{align}
\tilde{v}(\omega) = v^*(\omega), \\
\tilde{U}_\omega = U^*_*(\omega),
\end{align}
(i.e., the Komlos limit and the weak star limit are equal almost everywhere).

Gathering the sets of measure zero, let
\begin{equation}
N := N^\infty \cup \tilde{N} \cup N^*.
\end{equation}

We will apply Komlos’ Theorem a second time to establish the following:
CLAIM 1: There exists a measurable selection, $\omega \rightarrow \overline{U}(\omega)$, of the correspondence, $\omega \rightarrow \mathcal{L}^\infty \overline{U}(\omega)$, such that $\overline{U}(\omega) = \tilde{U}_\omega$ for all $\omega \in \Omega \setminus N$.

Proof of claim 1: Fix $\omega \in \Omega \setminus N$ for the moment and observe that the sequence of functions, $\{\overline{U}(\omega)\}_n$, is mean norm bounded, where recall
\begin{equation}
\omega' \rightarrow \overline{U}(\omega') := \left(1 - \beta_d r_d(\omega, \mu^\omega) + \beta_d \mathcal{Y}_d(\omega') \rho(\omega', \mu^\omega)\right)_{d \in D}.
\end{equation}
Applying Komlos’ Theorem to the sequence, $\{\overline{U}(\omega)\}_n$, we will assume WLOG that the sequence, $\{\mathcal{U}(\omega)\}_n$, itself $K$-converges to an integrable function, $\overline{U}(\omega)$. Thus, off a set, $\tilde{N}_\omega$, of $(\lambda \times \mu)$-measure zero we have,
\begin{equation}
\mathcal{U}(\omega') := \frac{1}{m'} \sum_{k=1}^{m'} \mathcal{U}^k(\omega') \overset{R_m}{\longrightarrow} \mathcal{U}(\omega') \text{ for all } \omega' \in \Omega \setminus \tilde{N}_\omega,
\end{equation}
Moreover, by Komlos Theorem we know that for any further subsequence $\{\mathcal{U}(\omega)\}_n'$, the corresponding sequence of arithmetic means converges pointwise to the same $K$-limit, $\mathcal{U}(\omega)$, but perhaps off of a different subset of $(\lambda \times \mu)$-measure zero. We have therefore, for some possibly new set of measure zero, $\tilde{N}_\omega'$, that
\begin{equation}
\mathcal{U}(\omega') := \frac{1}{m'} \sum_{k=1}^{m'} \mathcal{U}^k(\omega') \overset{R_m}{\longrightarrow} \mathcal{U}(\omega') \text{ for all } \omega' \in \Omega \setminus \tilde{N}_\omega'.
\end{equation}

Going back to our original $K$-limit, $\tilde{U}(\cdot)$, by Page (1991) Proposition 1, we have
\begin{equation}
\tilde{U}_\omega \in \text{co} \mathcal{L}^\infty \{\mathcal{U}(\omega)\} \text{ for all } \omega \in \Omega \setminus N.
\end{equation}
By theorem 8.2 in Wagner (1977), because $\tilde{U}_\omega \in \text{co} \mathcal{L}^\infty \{\mathcal{U}(\omega)\}$, the $K$-limit function, $\omega \rightarrow \tilde{U}_\omega$, has a Caratheodory representation given by
\begin{equation}
\omega \rightarrow \tilde{U}_\omega = \sum_{i=0}^{m} \alpha^i(\omega) \mathcal{U}^i.
\end{equation}
where the \( m + 1 \) functions, \( \{\alpha^0(\omega), \alpha^1(\omega), \ldots, \alpha^m(\omega)\} \) take nonnegative values and

\[
\sum_{i=0}^{m} \alpha^i(\omega) = 1
\]

for all \( \omega \), and where for each \( i = 0, 1, \ldots, m \), \( \overline{U}^{i}_{\omega} \in Ls\{\Upsilon^{i}_{\omega}\} \) for all \( \omega \in \Omega \). Because \( \overline{U}^{i}_{\omega} \in Ls\{\Upsilon^{n}_{\omega}\} \) there is some subsequence, \( \{\Upsilon^{i}_{\omega}\}^n_{n'} \), such that

\[
\lim_{n_i'} \overline{U}^{i}_{\omega} = \Upsilon^{i}_{\omega}.
\]

Thus because \( \Upsilon^{i}_{\omega} = \int_{\Omega} \Upsilon^{i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') \), we have

\[
\lim_{n_i'} \int_{\Omega} \Upsilon^{i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') = \Upsilon^{i}_{\omega}.
\quad (71)
\]

Also, because \( \lim_{n'_i} \int_{\Omega} \Upsilon^{i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') = \Upsilon^{i}_{\omega} \), we have by the properties of convergence of arithmetic means that

\[
\begin{aligned}
\lim_{n'_i} \frac{1}{n'_i} \sum_{n'=1}^{n'_i} \Upsilon^{i}_{\omega} & = \lim_{n'_i} \frac{1}{n'_i} \sum_{n'=1}^{n'_i} \int_{\Omega} \Upsilon^{i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') \\
& = \int_{\Omega} \lim_{n'_i} \frac{1}{n'_i} \sum_{n'=1}^{n'_i} \Upsilon^{i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') \\
& = \Upsilon^{i}_{\omega}.
\end{aligned}
\quad (72)
\]

Because the original sequence \( \{\Upsilon^{i}_{\omega}(\cdot)\}_n \) \( K \)-converges to \( K \)-limit \( \hat{\Upsilon}_{\omega}(\cdot) \), by Komlos’ Theorem we know that the subsequence \( \{\Upsilon^{n'_i}_{\omega}(\cdot)\}_{n'_i} \), \( K \)-converges to the same \( K \)-limit, \( \hat{\Upsilon}_{\omega}(\cdot) \) but perhaps off of a different subset of \( (\lambda \times \mu) \)-measure zero for each \( i \). Thus, for some set, \( \hat{N}^{i}_{\omega} \), possibly different from \( \hat{N}_{\omega} \), we have

\[
\hat{\Upsilon}^{n'_i}_{\omega}(\omega') := \frac{1}{n'_i} \sum_{n'=1}^{n'_i} \Upsilon^{n'_i}_{\omega}(\omega') \rightarrow \hat{\Upsilon}_{\omega}(\omega') \text{ for all } \omega' \in \Omega \setminus \hat{N}^{i}_{\omega}.
\quad (73)
\]

Therefore we have

\[
\begin{aligned}
\int_{\Omega} \lim_{n'_i} \frac{1}{n'_i} \sum_{n'=1}^{n'_i} \Upsilon^{n'_i}_{\omega}(\omega')d(\lambda \times \mu)(\omega') & = \int_{\Omega} \hat{\Upsilon}_{\omega}(\omega')d(\lambda \times \mu)(\omega') \\
& = \Upsilon^{i}_{\omega}.
\end{aligned}
\quad (74)
\]

Thus, \( \Upsilon^{0}_{\omega} = \Upsilon^{1}_{\omega} = \cdots = \Upsilon^{m}_{\omega} := \Upsilon_{\omega} \). But we also have

\[
\begin{aligned}
\hat{U}_{\omega} & = \sum_{i=0}^{m} \alpha^i(\omega)\Upsilon^{i}_{\omega} \\
& = \sum_{i=0}^{m} \alpha^i(\omega)\Upsilon^{i}_{\omega} \\
& = \sum_{i=0}^{m} \alpha^i(\omega)\int_{\Omega} \hat{\Upsilon}_{\omega}(\omega')d(\lambda \times \mu)(\omega') \\
& = \int_{\Omega} \hat{\Upsilon}_{\omega}(\omega')d(\lambda \times \mu)(\omega').
\end{aligned}
\quad (75)
\]
Thus, for this fixed $\omega \in \Omega\setminus N$, we have $\overline{U}_\omega = \hat{U}_\omega$.

By Theorem 7.1 in Himmelberg (1971), we have a measurable selection, $\omega \rightarrow \overline{\pi}_\omega^*$, from the Nash mapping, $\omega \rightarrow \mathcal{N}(\omega; \nu^*)$, such that

$$
\overline{U}_\omega = ((1 - \beta_d) \rho_d(\omega, \overline{\pi}_\omega^*) + \beta_d \int_{\Omega} \overline{\pi}_\omega^* h(\omega'|\omega, \overline{\pi}_\omega^*) d(\lambda \times \mu)(\omega'))_{d \in D}.
$$

Finally, observe that $\overline{U}_\omega^* \in Ls(\overline{U}_\omega)$, where recall that $\{ (\overline{\pi}_\omega^*, \overline{U}_\omega^*) \}_{\omega \in \Omega\setminus N}$ is a sequence of selections such that for all $n$ and for all $\omega \in \Omega\setminus N$, $(\lambda \times \mu)(N) = 0$

$$
\rho_{\omega^*}(v^n, \overline{U}_\omega^*) + \rho_X(v^n(\omega), \overline{U}_\omega^*) < \frac{1}{n}.
$$

Because this construction holds state-by-state for each $\omega \in \Omega\setminus N$, we have $\overline{U}_\omega^* = \hat{U}_\omega$ and therefore,

$$
(\nu^*, \hat{U}_\omega) = (\nu^*, \overline{U}_\omega) \in Gr\mathcal{P}(\omega, \cdot) \text{ for all } \omega \in \Omega\setminus N.
$$

Because $\nu^*(\omega) = \hat{v}(\omega)$ for all $\omega \in \Omega\setminus N$, the proof will be nearly complete if we can show that $\hat{v}(\omega) = \tilde{U}_\omega$ for all $\omega \in \Omega\setminus N$.

**CLAIM 2:** $\hat{v}(\omega) = \tilde{U}_\omega$ for all $\omega \in \Omega\setminus N$.

**Proof of CLAIM 2:** We have for all $\omega \in \Omega\setminus N$,

$$
\rho_X(\hat{v}(\omega), \tilde{U}_\omega) \\
\leq \rho_X(\hat{v}(\omega), \tilde{v}_n(\omega)) + \rho_X(\tilde{v}_n(\omega), \overline{U}_\omega^*) + \rho_X(\overline{U}_\omega^*, \tilde{U}_\omega).
$$

(77)

Because $\{ v^n(\cdot), \overline{U}_\omega^* \}_{\omega \in \Omega\setminus N}$, $K$-converges to $(\tilde{v}(\cdot), \tilde{U}_\omega)$ in $\mathcal{L}_X \times \mathcal{L}_X$, it is easy to see that in expression (77), 1 and 3 converge to zero. To see that 2 converges to zero, observe that for all $\omega \in \Omega\setminus N$,

$$
\rho_X(\hat{v}(\omega), \overline{U}_\omega^*) \\
= \rho_X\left( \frac{1}{n} \sum_{k=1}^n v^k(\omega), \frac{1}{n} \sum_{k=1}^n \overline{U}_\omega^k \right) \\
\leq \frac{1}{n} \sum_{k=1}^n \rho_X(v^k(\omega), \overline{U}_\omega^k).
$$

(78)

Therefore, Part C of expression (59) implies that for all $\omega \in \Omega\setminus N$, $\rho_X(v^n(\omega), \overline{U}_\omega^*) \rightarrow 0$, and therefore, for all $\omega \in \Omega\setminus N$

$$
v^*(\omega) = \hat{v}(\omega) = \tilde{U}_\omega = \overline{U}_\omega.
$$

To complete the proof, let $\overline{U}_\omega^2$ be any everywhere measurable selection from $\omega \rightarrow Ls\{ \overline{U}_\omega^* \}$ (i.e., $\overline{U}_\omega^2 \in Ls\{ \overline{U}_\omega^* \}$ for all $\omega$). Define the following measurable functions:

$$
U_\omega^* = \overline{U}_\omega^* I_{\Omega\setminus N}(\omega) + \overline{U}_\omega^2 I_N(\omega)
$$

and

$$
v^*(\omega) = \overline{v}(\omega) I_{\Omega\setminus N}(\omega) + \overline{U}_\omega^2 I_N(\omega).
$$

We have $v^n \rightarrow v^*$ and $v^*(\omega) = U_\omega^* \in Ls\{ \overline{U}_\omega^* \}$ for all $\omega \in \Omega$. Therefore,

$$
v^*(\omega) = U_\omega^* \in \mathcal{P}(\omega, v^*) \subseteq X \text{ for all } \omega.
$$

Finally, given that

$$
\mathcal{P}(\omega, v^*) := \{ U \in X : U = U((\omega, v^*), a) \text{ for some } a \in \mathcal{N}(\omega, v^*) \},
$$

35
where recall $U((\omega, v^*), a) := (u_d(\omega, a)(v^*_d))_{d \in D}$, we have by Theorem 7.1 in Himmelberg (1971) the existence of a measurable selection, $f^*(\cdot)$ of the $v^*$-Nash correspondence, $\omega \rightarrow N(\omega, v^*)$, such that

$$v^*(\omega) = U((\omega, v^*), f^*(\omega)) := (u_d(\omega, f^*(\omega))(v^*_d))_{d \in D} \text{ for all } \omega \in \Omega.$$ 

For the convenience of the reader we state the definitions of $K$-convergence and mean norm boundedness, as well as the Theorems of Komlos (1967) and Page (1991).

Definitions

(K-Convergence for $R^m$-Valued Functions) A sequence $\{v^n(\cdot)\}_n$ of $R^m$-valued functions, $K$-converges to an $R^m$-valued function $\hat{v}(\cdot)$ if and only if

$$\frac{1}{n} \sum_{k=1}^{n} v^k(\omega) \rightarrow \hat{v}(\omega) \text{ a.e. } [(\lambda \times \mu)],$$

and the same is true for any further subsequence. (79)

(Mean Norm Boundedness for $R^m$-Valued Functions)

A sequence, $\{v^n(\cdot)\}_n$, of $R^m$-valued functions is mean norm-bounded provided

$$\sup_n \int_{\Omega} \|v^n(\omega)\|_{R^m} d(\lambda \times \mu)(\omega) < \infty.$$ 

Theorems

 Komlos’ Theorem, 1967) If the sequence, $\{v^n(\cdot)\}_n$, of $R^m$-valued functions is mean norm-bounded, then $\{v^n(\cdot)\}_n$ has a subsequence $K$-converging to some integrable $R^m$-valued function, $\hat{v}(\cdot)$.

Page’s Theorem, 1991) If the sequence, $\{v^n(\cdot)\}_n$, of $R^m$-valued functions is mean norm-bounded and $K$-converges to some integrable $R^m$-valued function, $\hat{v}(\cdot)$, then

$$\hat{v}(\omega) \in \text{coLs}\{v^n(\omega)\} \text{ a.e.}$$

and there exists an integrable $R^m$-valued function, $v^*(\cdot)$, such that $v^*(\omega) \in \text{Ls}\{v^n(\omega)\}$ a.e. and

$$\int_{\Omega} v^*(\omega) d(\lambda \times \mu)(\omega) = \int_{\Omega} \hat{v}(\omega) d(\lambda \times \mu)(\omega).$$

7 Risky Connections and Approximability

We have shown that all approximable discounted stochastic games of network formation have stationary Markov equilibria. Our next objective is to show that all discounted stochastic games of network formation with risky connections satisfying assumptions [A-1] and [A-2] are approximable.

Theorem 6 (All Discounted Stochastic Games of Network Formation with Risky Connections Are Approximable)

All discounted stochastic game of network formation with risky connections satisfying assumptions [A-1] and [A-2] are approximable.
Proof. Let \( NDSG \) be any discounted stochastic game of network formation with risky connections satisfying assumptions \([A-1]\) and \([A-2]\) and let \( \{G_\omega, \mathcal{X}_\omega, \mathcal{N}(\omega, \cdot), \mathcal{P}(\omega, \cdot)\} \) be the corresponding one-shot triple.

Given current state and value function profile pair, \((\omega, v) \in \Omega \times \mathcal{L}_\mathcal{X}^\infty\), the \( \omega \)-Nash payoffs is given by

\[
\mathcal{P}(\omega, v) := \int_{\mathcal{G}} \mathcal{P}(\omega, v)(\bar{G}') \lambda(d\bar{G}'),
\]

where in each risky state \( \bar{G}' \) the nonempty, closed subset, \( \mathcal{P}(\omega, v)(\bar{G}') \subset X \), of Nash payoffs given by

\[
\mathcal{P}(\omega, v)(\bar{G}') := \left\{ \left. U_d \in X_d \right| U_d = (1 - \beta_d) r_d(\omega, \sigma) + \beta_d V_d(\bar{G}', \omega, \sigma) \text{ for some } \sigma \in \mathcal{N}(\omega, v) \right\}_{d \in D},
\]

where

\[
V_d(\bar{G}', \omega, \sigma) := \int_{\mathcal{G} \times \mathcal{F}} v_d(\bar{G}', (G', S')) h_{x}(\bar{G}'|G', S') \rho(d(G', S')|\omega, \sigma)
\]

and where \( h_x(\cdot|G', S') \) is the density of the probability measure \( \varepsilon(d\bar{G}'|G', S') \) on risky states with respect to the nonatomic measure \( \lambda(d\bar{G}') \) given regular state \((G', S')\).

Given that for each \((\omega, v)\),

\[
\bar{G}' \to \mathcal{P}(\omega, v)(\bar{G}'),
\]

is a \((\mathcal{B}(\mathcal{G}), \mathcal{B}(X))\)-measurable set-valued mapping defined on the nonatomic probability space \((\mathcal{G}, \mathcal{B}(\mathcal{G}), \lambda)\), taking nonempty, closed values in \( X \), if we take the Aumann integral of the set-valued mapping, \( \mathcal{P}(\omega, v)(\cdot) \), we have

\[
\mathcal{P}(\omega, v) := \int_{\mathcal{G}} \mathcal{P}(\omega, v)(\bar{G}') \lambda(d\bar{G}').
\]

Because the probability space \((\mathcal{G}, \mathcal{B}(\mathcal{G}), \lambda)\) is nonatomic, by Lyapunov’s Theorem (1940), the \( \omega \)-Nash payoff USCO \( v \to \mathcal{P}(\omega, v) \) is convex valued. Thus, by Theorem 1 in Cellina (1969), for each current state \( \omega \), the \( \omega \)-Nash payoff USCO, \( \mathcal{P}(\omega, \cdot) \) is approximable and we can conclude that any discounted stochastic game of network formation with risky connections is approximable. □
8 Endogenous Network Dynamics

We are now in a position to analyze the stability properties of a strategically informed, risky process of network and coalition formation.

8.1 Equilibrium Transitions

Under the equilibrium stationary Markov strategy, $\sigma^*(\cdot)$, the Markov process of network and coalition formation,

$$\left\lbrace \mathcal{W}_n^* \right\rbrace_n = \left\lbrace (\mathcal{G}_n^*, (\mathcal{G}_n^*, S_n^*)) \right\rbrace_{n=1}^\infty,$$

is governed by the equilibrium Markov transition,

$$p^*(E|\omega) := q(E|\omega, \sigma^*(\omega))$$

$$= \int_{\mathcal{G}_n^*} q(E|\omega, \mathcal{G}_n^*) \sigma^*(d\mathcal{G}_n^*|\omega).$$

Thus,

$$\Pr \left\lbrace \mathcal{W}_{n+1}^* \in E|\mathcal{W}_n^* = \omega \right\rbrace = p^*(E|\omega)$$

and

$$\Pr \left\lbrace \mathcal{W}_n^* \in E|\mathcal{W}_0^* = \omega \right\rbrace = p^{*n}(E|\omega) = q^n(E|\omega, \sigma^*(\omega)),$$

where the $n$-step transition $p^{*n}(\cdot|\cdot)$ is defined recursively as follows: for all $\omega \in \Omega$ and $E \in B(\Omega)$,

$$p^{*n}(E|\omega) = \int_\Omega p^*(E|\omega') p^{*n-1}(d\omega'|\omega) = \int_\Omega p^{*n-1}(E|\omega') p^*(d\omega'|\omega),$$

for $n = 1, 2, \ldots$, and $p^{*0}(\cdot|\omega) = \delta_\omega(\cdot)$ is the Dirac measure at $\omega$.

8.2 Endogenous Absorbing Sets and Invariant and Ergodic Probability Measures

A set $E \in B(\Omega)$ (of network-network-coalition 3-tuples) is called a $p^*$-absorbing set if $p^*(E|\omega) = 1$ for all network-network-coalition 3-tuples $\omega \in E$. Let $\mathcal{L}^* \subseteq B(\Omega)$ denote the collection of all $p^*$-absorbing sets. A $p^*$-absorbing set $E \in \mathcal{L}^*$ is said to be indecomposable if it does not contain the union of two disjoint absorbing sets. Note that the set of all absorbing sets is closed under countable unions and intersections.

A probability measure $\gamma(\cdot)$ on the state space of feasible network-network-coalition 3-tuples $(\Omega, B(\Omega))$ is invariant for Markov transition $p^*(\cdot|\cdot)$ (i.e., is $p^*$-invariant) if

$$\gamma(E) = \int_\Omega p^*(E|\omega) d\gamma(\omega)$$

for all $E \in B(\Omega)$.

Thus, if probability measure $\gamma(\cdot)$ is $p^*$-invariant, then for any set of network-network-coalition 3-tuples, $E \in B(\Omega)$, if the status quo state $\omega^n = (\mathcal{G}_n, G_n, S_n)$ is chosen according to probability measure $\gamma(\cdot)$ - so that the probability that $\omega^n$ lies in $E$ is just $\gamma(E)$ - then the probability that next period’s state $\omega^{n+1} = (\mathcal{G}_{n+1}, G_{n+1}, S_{n+1})$ lies in $E$ is also $\gamma(E) = \int_\Omega p^*(E|\omega) d\gamma(\omega)$. Denote by $\mathcal{T}^*$ the collection of all $p^*$-invariant measure.

A $p^*$-invariant measure $\gamma(\cdot)$ is said to be $p^*$-ergodic if $\gamma(E) = 0$ or $\gamma(E) = 1$ for all $E \in \mathcal{L}^*$. Denote by $\mathcal{E}^*$ the collection of all $p^*$-ergodic measures. Because the $p^*$-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{T}^*$ of $p^*$-invariant measures (see Theorem 19.25 in Aliprantis and Border 2006), each measure $\gamma(\cdot)$ in $\mathcal{T}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.
8.3 Visitations and Hitting Times

The number of visitations by the process \( \{ W_n^* \} = \{(G_n^*, (G_n^*, S_n^*))\}_{n=1}^\infty \) to the set of network-network-coalition 3-tuples \( E \in B(\Omega) \), is given by

\[
\eta^*_E := \sum_{n=1}^{\infty} I_E(W_n^*),
\]

where \( I_E(W_n^*) = 1 \) if \( W_n^* \in E \) and zero otherwise. Thus, the expected number of visitations to \( E \) starting from network-network-coalition 3-tuple \( \omega = (e, G, S) \) is given by

\[
G^*(\omega, E) := E^*_\omega[\eta^*_E] = \sum_{n=1}^{\infty} p^n(E|\omega).
\]

The probability that the network-coalition formation process \( \{ W_n^* \} \), visits \( E \) infinitely often (denoted by i.o.) is given by

\[
Q^*(\omega, E) := \Pr\{ \lim_{n \to \infty} W_n^* = \omega \} = \Pr\{ \eta^*_E = \infty | W_0^* = \omega \}
\]

for all \( \omega \in \Omega \). The hitting time for set \( E \) is given by

\[
\tau^*_E := \inf \left\{ n \geq 1 : W_n^* \in E \right\}.
\]

Following Tweedie (2001),

\[
L^*(\omega, E) := \Pr\{ \tau^*_E < \infty | W_0^* = \omega \} = \Pr\{ \bigcup_{n=1}^{\infty} (W_n^* \in E | W_0^* = \omega) \}
\]

is the probability that the process \( \{ W_n^* \} \) hits (or reaches) in finite time the set of network-network-coalition 3-tuples \( E \) starting from network-network-coalition 3-tuple \( \omega \in \Omega \) given transition \( p^*(\cdot | \cdot) \). By Proposition 9.1.1 in Meyn and Tweedie (2009), if for any \( E \in B(\Omega), L^*(\omega, E) = 1 \) for all \( \omega \in E \), then

\[
L^*(\omega, E) = Q^*(\omega, E) \quad \text{for all} \quad \omega \in \Omega.
\]

8.4 Recurrence, Transience, and Irreducibility

The set of network-network-coalition 3-tuples \( E \) is recurrent if

\[
G^*(\omega, E) := E^*_\omega[\eta^*_E] = \sum_{n=1}^{\infty} p^n(E|\omega) = +\infty.
\]

By Proposition 8.1.3 in Meyn and Tweedie (2009), for any state \( \omega \in \Omega \),

\[
G^*(\omega, \{\omega\}) = +\infty \quad \text{if and only if} \quad L^*(\omega, \{\omega\}) = 1.
\]

A set of network-network-coalition 3-tuples \( T \in B(\Omega) \) is transient if (i) \( T \) is the disjoint union of countably many uniformly transient sets \( U_j \), that is, sets \( U_j \in B(\Omega) \) such that
\( T = \cup_j U_j \) and if (ii) for each set there is a finite constant \( M_j \), such that for all network-network-coalition 3-tuples \( \omega \in U_j \),

\[
E^*_\omega[\eta^{n_j}] = \sum_{n=1}^{\infty} p^{*n}(U_j | \omega) < M_j. \tag{88}
\]

The set of network-network-coalition 3-tuples \( E \) is said to be \( p^* \)-inessential if

\[
Q^*(\omega, E) = 0 \text{ for all } \omega \in \Omega. \tag{89}
\]

Thus, a set of states \( E \) is inessential if the probability that the network-coalition formation process visits the set \( E \) infinitely often is zero stating from any state. If a set of states is inessential, then if the process visits the state at all, it leaves the state for good after finitely many moves. The union of countable many inessential states is called an inproperly \( p^* \)-essential set. Any other set is called properly \( p^* \)-essential.

Finally, the network-coalition formation process \( \{\tilde{W}^*_n\}_n \), governed by \( p^*(\cdot | \cdot) \) is said to be \( \psi \)-irreducible if for some probability measure \( \psi(\cdot) \) on \( B(\Omega) \),

\[
\psi(E) > 0 \text{ implies } L^*(\omega, E) > 0 \text{ for all } \omega \in \Omega.
\]

Thus if the process \( \{\tilde{W}^*_n\}_n \) governed by \( p^*(\cdot | \cdot) \) is \( \psi \)-irreducible, then it hits all the “important” sets of network-network-coalition 3-tuples (i.e., the sets \( E \in B(\Omega) \) such that \( \psi(E) > 0 \)) with positive probability starting from any network-network-coalition 3-tuple in the state space \( \Omega = G \times (G \times F) \).

The network-coalition formation process \( \{\tilde{W}^*_n\}_n \), governed by \( p^*(\cdot | \cdot) \) is said to be \( \psi \)-recurrent if,

\[
\psi(E) > 0 \text{ implies } Q^*(\omega, E) = 1 \text{ for all } \omega \in \Omega.
\]

9 Stability of Endogenous Network Dynamics

In addition to modeling the emergence of endogenous network dynamics from the co-evolution of strategic behavior in forming risky connections and network structure, one of our main objectives is to study the dynamic stability properties of the resulting equilibrium process of network and coalition formation. A key component of our analysis is the notion of a dynamic basin of attraction. Intuitively, a set of network-network-coalition 3-tuples \( H \) is a basin of attraction if the network and coalition formation process \( \{\tilde{W}^*_n\}_n \) reaches \( H \) in finite time with probability 1 and once there, stays there. The question we wish to answer is this: does the process of network and coalition formation \( \{\tilde{W}^*_n\}_n \), that emerges from the equilibrium interplay of strategic behavior, network structure, and the trembles of nature (both regular and risky) generate basins of attraction. We begin by considering the classical notion of a Maximal Harris set of network and coalition pairs.

9.1 Dynamic Strategic Basins of Attraction: Maximal Harris Sets

A set of network-network-coalition 3-tuples \( H \in B(\Omega) \) is called a maximal Harris set if there exists some probability measure \( \varphi(\cdot) \) on \( B(\Omega) \) such that \( \varphi(H) > 0 \),

\[
\varphi(A) > 0 \text{ implies } L^*(\omega, A) = 1 \text{ for all } \omega \in H, \tag{15}
\]

and

\[
L^*(\omega, H) = 1 \text{ implies that } \omega \in H.
\]

\[15\] Here, the probability measure \( \psi(\cdot) \) is a maximal irreducibility measure (see Section 4.2.2 in Meyn and Tweedie, second edition, 2009).
Note that a maximal Harris set is a maximal absorbing set and is indecomposable. Moreover, if $H$ and $H'$ are distinct Maximal Harris sets, then they are disjoint. Finally, note that if the network-coalition formation process reaches a particular Harris set then it remains there for all future periods. By Proposition 9.1.1 in Meyn and Tweedie (2009), because we have $L^*(\omega, H) = 1$ for all $\omega \in H$,$$
abla^*\omega, H = Q^*(\omega, H) = 1$$for all $\omega \in H$.

Thus, if the set of network-network-coalition 3-tuples $H$ is maximal Harris, then process $\{W_t\}_n$ restricted to $H$ is $\varphi$-irreducible and Harris recurrent - where Harris recurrence means that $Q^*(\omega, E) = 1$ for all $\omega \in H$.

The fact that a maximal Harris set is a maximal absorbing set makes it a good candidate for a basin of attraction. But in order to fully qualify as a basin of attraction we must show that - or identify conditions under which - the process reaches such a set in finite time with probability 1.

9.2 The Fundamental Conditions for Stability: Drift and Global Uniform Countable Additivity

Given the equilibrium Markov transition $p^*(\cdot|\cdot)$ what can be said concerning stability? What conditions guarantee that the equilibrium process of network and coalition formation reaches a Harris set in finite time with probability 1. It turns out that the Tweedie Conditions (2001) do just that:

The Tweedie Conditions (2001):

there exists a measurable set of network-network-coalition 3-tuples $C \subseteq \Omega$, a nonnegative measurable function

$V(\cdot) : \Omega \rightarrow [0, \infty]$,

and a finite real number $b$ such that

(i) (the drift condition) for all $\omega \in \Omega$

\[
\int_\Omega V(\omega')dp^*(\omega'|\omega) \leq V(\omega) - 1 + bI_C(\omega),
\]

and

(ii) (uniform countable additivity) for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

\[
\lim_{n \rightarrow \infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0.
\]

We say that the Markov transition $p^*(\cdot|\cdot)$ satisfies global uniform countable additivity if for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

\[
\lim_{n \rightarrow \infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0,
\]

and we will say that the Tweedie conditions are satisfied globally if both conditions (i) and (ii) hold with $C = \Omega$.

Using results due to Meyn and Tweedie (2009), Tweedie (2001), and Costa and Dufour (2005), we will show below that if the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the equilibrium process of network and coalition formation is globally uniformly countably
additive, then the equilibrium process possesses some striking stability properties - analogous to those demonstrated in Page and Wooders (2009a) for static abstract games of network formation.

To begin, let us strengthen our assumptions [A-2](6) concerning the law of motion by adding to the list assumption [A-2](6)(iv).

[A-2](6)(iv) The collection of probability densities $H_{(\lambda \times \mu)}$ is bounded by a $(\lambda \times \mu)$-integrable function, $g(\cdot) : \Omega \to R_+$.  

By [A-2](6)(iv), we have for all $h(\cdot|\omega, G_D) \in H_{(\lambda \times \mu)}$, 

$$0 \leq h(\omega'|\omega, G_D) \leq g(\omega') \text{ a.e.}[(\lambda \times \mu)] \text{ in } \omega'.$$

Denote by [A-2]∗ our augmented list of assumptions about the discounted stochastic game of network formation. We have our main result on global uniform countable additivity.

**Theorem 7 (Global Uniform Countable Additivity)**

Suppose assumptions [A-1], [A-2] and [A-4]∗ hold. Then $p^*(\cdot|\cdot)$ is globally uniformly countably additive.

**Proof.** For any sequence $B_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$p^*(B_n|\omega) = \int_{B_n} q(\omega'|\omega, \sigma^*(\omega)) = \int_{G^m} \int_{B_n} q(\omega'|\omega, G_D') \sigma^*(dG_D'|\omega)) = \int_{G^m} \left( \int_{B_n} h(\omega'|\omega, G_D') d(\lambda \times \mu)(\omega') \right) \sigma^*(dG_D'|\omega)) \leq \int_{B_n} g(\omega') d(\lambda \times \mu)(\omega') \to 0 \text{ as } B_n \downarrow \emptyset,$$

where $g(\cdot)$ is the $(\lambda \times \mu)$-integrable function bounding the set of densities $H_{(\lambda \times \mu)}$. □

Under assumptions [A-1] and [A-2]∗ the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of network and coalition formation is globally uniformly countably additive. Moreover, letting $C = \Omega$, $V(\omega) = 1$ for all $\omega \in \Omega$, and $b = 2$, the drift condition is also satisfied. Thus, by strengthening the stochastic continuity properties of the law of motion $q(\cdot|\cdot, \cdot)$ mildly beyond what is required to guarantee the existence of an equilibrium stationary Markov transition, $p^*(\cdot|\cdot)$, we are able to conclude that the Tweedie conditions are satisfied globally (i.e., with $C = \Omega$).

9.3 Strategic Basins of Attraction, Invariance, and Ergodicity

We now have our main result concerning stochastic basins of attraction and the stability of the equilibrium network-coalition formation process

$$\left\{ \tilde{W}_n^* \right\}_n = \left\{ (\tilde{G}_n, S_n) \right\}_n^{\infty}$$

governed by $p^*(\cdot|\cdot)$.  

42
Theorem 8 (Basins of Attraction: The Finite Decomposition of the State Space)

Under assumptions [A-1] and [A-2]* the equilibrium network-coalition formation process
\[
\left\{ \bar{W}_n^* \right\}_n = \left\{ (\bar{G}_n^*, (G_n^*, S_n^*)) \right\}_{n=1}^{\infty}
\]
governed by the equilibrium Markov transition \( p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot)) \) generates a decomposition of the state space of network-network-coalition 3-tuples \( \Omega = \mathbb{G} \times (\mathbb{G} \times \mathcal{F}) \) into a finite number of disjoint basins of attraction and a disjoint transient set. In particular, this decomposition is of the form
\[
\Omega = \left( \bigcup_{i=1}^{N} H_i \right) \cup T, \quad (91)
\]
where each \( H_i \) is a basin of attraction (i.e., maximal Harris) and \( T \) is transient, and has the property that for every network-network-coalition 3-tuple \( \omega \in \Omega \)
\[
L^* (\omega, \bigcup_{i=1}^{N} H_i) = 1. \quad (92)
\]

By Theorem 8 the equilibrium network-coalition formation process \( \left\{ \bar{W}_n^* \right\}_n \) is such that starting at any network-network-coalition 3-tuple not contained in a basin of attraction (i.e., a maximal Harris set), the process will reach in finite time with probability 1, one of finitely many basins of attraction \( H_i \), and once there will stay there. An analogous conclusion is reached in Page and Wooders (2009a) for static, abstract games of network formation over finitely many networks. There it is shown that no matter what rules of network formation prevail, given any profile of player preferences, the feasible set of networks contains a finite, disjoint collection of sets, each set representing a strategic basin of attraction in the sense that if the game is repeated - each time starting at the status quo network reached in the previous play of the game - the process of network formation generated by repeating this static game will reach a strategic basin of attraction in finitely many moves and once there will stay there.

Because in our model the Tweedie conditions hold globally, it follows from Theorem 2 in Tweedie (2001) that the entire state space \( \Omega \) admits a finite decomposition,
\[
\Omega = \left( \bigcup_{i=1}^{N} H_i \right) \cup T,
\]
consisting of a finite number of indecomposable, Maximal Harris sets, \( H_i \), and a transient set \( T \). The key step in establishing this finite decomposition is to show that because the equilibrium Markov transition,
\[
\omega \to q(\cdot|\omega, \sigma^*(\omega)),
\]
is globally, uniformly countably additive, the state space contains at most a finite number of disjoint absorbing sets (see Tweedie 2001, Lemma 2). Moreover, by Theorem 2 in Tweedie (2001), this decomposition is such that \( L^* (\omega, \bigcup_{i=1}^{N} H_i) = 1 \) for all \( \omega \in \Omega \). Thus, governed by the equilibrium Markov transition, \( q(\cdot|\cdot, \sigma^*(\cdot)) \), the process of network and coalition formation is such that no matter where the process begins (no matter what network-network-coalition 3-tuple is the starting point), it reaches in finite time with probability 1 one of finitely many basins of attraction, \( H_i \), and once there, stays there. Thus, the proof of our Theorem 8 follows from Theorem 2 in Tweedie (2001) and the fact that the equilibrium Markov transition, \( q(\cdot|\cdot, \sigma^*(\cdot)) \), is globally uniformly countably additive.

Our next result establishes that the equilibrium Markov transition possesses a finite number of ergodic measures, one for each basin of attraction.
Theorem 9 (Invariance and Ergodicity of Endogenous Network Dynamics)

Suppose assumptions [A-1] and [A-2]* hold. Let

\[ \{\tilde{W}_n^*\}_{n=1}^\infty = \{(G_n^*, (G_n^*, S_n^*))\}_{n=1}^\infty \]

be the equilibrium network-coalition formation process governed by the equilibrium Markov transition \( p^*(\cdot | \cdot) = q(\cdot | \cdot, \sigma^*(\cdot)) \), and let

\( \Omega = (\cup_{i=1}^N H_i) \cup T \),

be the corresponding finite decomposition into basins of attraction.

The following statements are true:

1. Corresponding to each basin of attraction \( H_i \), there is a unique \( p^*\)-invariant probability measure \( \gamma_i(\cdot) \) with \( \gamma_i(H_i) = 1 \). Moreover, for each network-network-coalition 3-tuple \( \omega = (G, G, S) \),

\[ p^{*(n)}(E|\omega) := \frac{1}{n} \sum_{k=1}^n p^k(E|\omega) \overset{\Delta}{=} \sum_{i=1}^N L^*(\omega, H_i)\gamma_i(E \cap H_i), \text{ for all } E \in B(\Omega), \quad (93) \]

where \( p^k(E|\omega) \) is defined recursively, see (80).

2. The set of all ergodic probability measures is given by

\[ \mathcal{E}^* = \{\gamma_i(\cdot)\}_{i=1}^N \].

Moreover, a probability measure \( \gamma(\cdot) \) on \((\Omega, B(\Omega))\) is \( p^*\)-invariant, i.e. \( \gamma(\cdot) \in \mathcal{I}^* \), if and only if \( \gamma(\cdot) \) is given by

\[ \gamma(E) = \sum_{i=1}^N \gamma(H_i)\gamma_i(E \cap H_i), \text{ for all } E \in B(\Omega). \]  

(94)

3. \( \mathcal{E}^* \) is a singleton (i.e., \( \mathcal{E}^* = \{\gamma(\cdot)\} \)) if and only if the network-coalition formation process \( \{\tilde{W}_n^*\}_{n=1}^\infty \) is \( \psi \)-irreducible, in which case for each network-network-coalition 3-tuple \( \omega = (G, G, S) \) and for every set of network-network-coalition 3-tuples \( E \in B(\Omega) \)

\[ \frac{1}{n} \sum_{k=1}^n p^k(E|\omega) \overset{\Delta}{=} \gamma(E). \]

Proof. (1) Under our assumptions [A-1] and [A-2]*, \( p^*(\cdot | \cdot) \) satisfies the Tweedie conditions globally. As a result, the first statement in part (1) is an immediate consequence of Lemma 5 in Tweedie (2001). The second statement also follows from the fact that in our model the Tweedie conditions hold globally and Theorem 1 in Tweedie (2001) (also, see Chapter 13 in Meyn and Tweedie 2009).

(2) Again because the Tweedie Conditions are satisfied globally, the first statement in part (2) follows from Lemma 2 in Tweedie (2001), Theorem 2.18 part (1) in Costa and Dufour (2005), Theorem 3.8 in Costa and Dufour, and the proof of Proposition 5.3 in Costa and Dufour. The second statement in part (2), that \( \gamma(\cdot) \in \mathcal{I}^* \) implies (94), follows from the proof of Proposition 5.3 in Costa and Dufour (2005). The fact that (94) implies \( \gamma(\cdot) \in \mathcal{I}^* \) follows from observation (but also, see Theorem 19.25 in Aliprantis and Border 2006 and Theorem 2 in Villarel 2004).

(3) Finally, because the Tweedie Conditions are satisfied globally, necessary and sufficient conditions for \( \mathcal{E}^* \) to be a singleton, given in terms of \( \psi \)-irreducibility follow from
Theorem 3 in Tweedie (2001). The convergence result in part (3) follows from the convergence result in part (1) of the Theorem and the fact that if there is only one basin of attraction \( H \) (i.e., one maximal Harris set), then by Theorem 3, \( L^*(\omega, H) = 1 \) for all \( \omega \in \Omega \). 

Note that the probability measures in \( \mathcal{E}^* \) are orthogonal, that is, for all \( i \) and \( i' \) in \( \{1, 2, \ldots, N\} \) with \( i \neq i' \),

\[
\gamma_i(\Omega \setminus H_i) = \gamma_i'(H_i) = 0.
\]

### 9.4 Ergodic Properties of Strategic Values

For each starting network-network-coalition 3-tuple \( \omega = (G, G, S) \in \Omega \), \( w^*_d(\omega) := \frac{\sigma^*_d(\omega)}{1 - \beta_d} \) is the strategic value to player \( d \) of following his stationary Markov strategy, \( \sigma^*_d(\cdot) \), given that all other players follow their strategy \( \sigma^*_d(\cdot) \). Because each Markov strategy profile \( \sigma^*(\cdot) \) is Nash, we know that this is the best that player \( d \) can do relative to all other strategies, even those that are history dependent. Strategies \( \sigma^*(\cdot) \) together with the trembles of nature (both regular and risky) determine the equilibrium Markov process of network and coalition formation via the transition \( p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot)) \). The questions we wish to address in this section concern the properties of players’ strategic values across time and states given the equilibrium process of network and coalition formation.

We begin by considering time averages. Let

\[
p^{*(n)} w^*_d(\omega) := \frac{1}{n} \sum_{k=1}^{n} \int_\Omega w^*_d(\omega') p^{*k}(d\omega'|\omega) = \int_\Omega w^*_d(\omega') p^{*(n)}(d\omega'|\omega),
\]

where recall,

\[
w^*_d(\omega) = E_d(\sigma^*(\omega)) := \sum_{n=1}^{\infty} \beta_d^{n-1} r^*_d(\sigma^*(\omega))
\]

\[
= r_d(\omega, \sigma^*(\omega)) + \beta_d \int_\Omega w^*_d(\omega') dq(\omega'|\omega, \sigma^*(\omega))
\]

and

\[
p^{*(n)}(E|\omega) := \frac{1}{n} \sum_{k=1}^{n} p^{*k}(E|\omega) = \frac{1}{n} \sum_{k=1}^{n} \int_\Omega p^*(E|\omega') p^{*k-1}(d\omega'|\omega).
\]

Here, \( p^{*k}(E|\omega) \) is the probability that process reaches the set of network-network-coalition 3-tuples \( E \) starting at network-network-coalition 3-tuple \( \omega = (G, G, S) \) in \( k \) periods or moves if each player follows his Markov strategy, \( \sigma^*_d(\omega) \).

The function \( p^{*(n)} w^*_d(\cdot) \) specifies for each starting network-network-coalition 3-tuple, player \( d \)’s \( n \)-period time average expected strategic value (i.e., the average value of following his stationary Markov strategy \( \sigma^*_d(\cdot) \) for \( n \) moves). We can think of \( \lim_{n} p^{*(n)} w^*_d(\cdot) \) therefore as specifying for each starting network-network-coalition 3-tuple, player \( d \)’s time average expected value.

By part (1) of Theorem 9 above, we have for all \( \omega \in \Omega \) and \( E \in B(\Omega) \)

\[
p^{*(n)}(E|\omega) = \frac{1}{n} \sum_{k=1}^{n} p^{*k}(E|\omega) \to \sum_{i=1}^{N} L^*(\omega, H_i) \gamma_i(E \cap H_i) = \gamma^*(E), \tag{95}
\]

where \( \gamma^*(\cdot) \in I^* \) for all \( \omega \in \Omega \) and \( I^* = \{\gamma_i(\cdot) : i = 1, 2, \ldots, N\} \). Because \( p^{*(n)}(\cdot|\omega) \) converges setwise for all \( \omega \), by Delbaen’s Lemma (1974) we have for all \( \omega \in \Omega \)

\[
p^{*(n)} w^*_d(\omega) \to \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w^*_d(\omega') d\gamma_i(\omega'). \tag{96}
\]

Thus, we obtain one of the fundamental principles of equilibrium dynamics: the equality of time averages and state averages.
Theorem 10 (The Equality of Time Average Values and State Average Values)

Under assumptions $[A-1]$ and $[A-2]$, the equilibrium network-coalition formation process is governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot,\sigma^*(\cdot))$ is such that:

1. Part (1) is an immediate consequence of part (1) of Theorem 4, Delbaen’s Lemma (1974), and the fact that for all basins of attraction $H_i$ for all initial states $\omega = (\bar{G},G,S) \in H_i$,

\[
\lim_{n \to \infty} p^{(n)}_i(\omega) = \frac{1}{|H_i|} \int_{H_i} w^*_i(\omega') d\gamma_i(\omega').
\]

Moreover, for all initial states $\omega = (\bar{G},G,S) \in \Omega$,

\[
\lim_{n \to \infty} p^{(n)}_i(\omega) = \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w^*_i(\omega') d\gamma_i(\omega').
\]

2. For all invariant measures $\gamma(\cdot) \in I^*$

\[
\int_{\Omega} f^*_i(\omega') d\gamma(\omega') = \int_{\Omega} \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w^*_i(\omega') d\gamma_i(\omega').
\]

where

\[
f^*_i(\omega) := \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w^*_i(\omega') d\gamma_i(\omega') \text{ for all } \omega \in \Omega.
\]

Proof. (1) Part (1) is an immediate consequence of part (1) of Theorem 4, Delbaen’s Lemma (1974), and the fact that for all basins $H_i$ and all states $\omega \in H_i$, $L^*(\omega, H_i) = 1$.

(2) Let invariant probability measure $\gamma(\cdot) = \sum_{i=1}^{N} \gamma(H_i) \gamma_i(\cdot) \in I^*$ be given. We have

\[
\int_{\Omega} w^*_i(\omega') d\gamma(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{\Omega} w^*_i(\omega') d\gamma_i(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} w^*_i(\omega') d\gamma_i(\omega'),
\]

and

\[
\int_{\Omega} f^*_i(\omega') d\gamma(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{\Omega} f^*_i(\omega') d\gamma_i(\omega') = \sum_{i=1}^{N} \gamma(H_i) \int_{H_i} f^*_i(\omega') d\gamma_i(\omega').
\]

Letting $\int_{H_i} f^*_i(\omega') d\gamma_i(\omega') := w^*_i(H_i)$, we have

\[
\int_{H_i} f^*_i(\omega') d\gamma_i(\omega') = \int_{H_i} \sum_{i=1}^{N} L^*(\omega', H_i) w^*_i(H_i) d\gamma_i(\omega').
\]

Moreover, because for all $\omega' \in H_i$, $L^*(\omega', H_i) = 1$ and $L^*(\omega', H_{i'}) = 0$, for all $i' \neq i$,

\[
\int_{H_i} \sum_{i=1}^{N} L^*(\omega', H_i) w^*_i(H_i) d\gamma_i(\omega') = w^*_i(H_i) = \int_{H_i} w^*_i(\omega') d\gamma_i(\omega').
\]
Thus we have for each $i$

$$\int_{H_i} f_d^*(\omega') d\gamma_i(\omega') = \int_{H_i} w_d^*(\omega') d\gamma_i(\omega'),$$

and thus,

$$\int_{\Omega} f_d^*(\omega') d\gamma(\omega') = \sum_{i=1}^N \gamma(H_i) \int_{H_i} f_d^*(\omega') d\gamma_i(\omega')$$

$$= \sum_{i=1}^N \gamma(H_i) \int_{H_i} w_d^*(\omega') d\gamma_i(\omega')$$

$$= \int_{\Omega} w_d^*(\omega') d\gamma(\omega').$$

The results above are essentially Birkhoff’s Ergodic Theorems (pointwise and mean) for equilibrium Markov network and coalition formation processes (see for example, Theorems 2.3.4 and 2.3.5 in Hernandez-Lerma and Lasserre 2003).

By part (1) of Theorem 10, each player’s time average value $\lim_n p^*(n) w_d^*(\omega) = f_d^*(\omega)$ is constant with respect to the starting network-network-coalition 3-tuple on each basin of attraction. In particular,

$$\lim_n p^*(n) w_d^*(\omega) = \int_{\Omega} w_d^*(\omega') d\gamma(\omega') = \int_{H_i} w_d^*(\omega') d\gamma_i(\omega') \text{ for all } \omega \in H_i.$$

By part (2) of Theorem 10, for any given invariant probability measure each player’s average of time averages over the entire state space is equal to his state average over the entire state space with respect to the given measure.
10 Strategic Stability and Dynamic Consistency

Let $\sigma^*(\cdot)$ be an equilibrium stationary Markov strategy profile of the dynamic network-coalition formation game over risky networks with corresponding equilibrium Markov transition $p^*(\cdot | \cdot) = q(\cdot | \cdot, \sigma^*(\cdot))$, and let

$$\Omega = \left( \bigcup_{i=1}^{N} H_i \right) \cup T,$$

be the finite decomposition of the state space generated by $p^*(\cdot | \cdot)$ with strategic basins of attraction $\{H_1, \ldots, H_N\}$ and transient set $T$. Each basin, $H_i$, is an absorbing set for the equilibrium state transition $p^*(\cdot | \cdot) := q(\cdot | \cdot, \sigma^*(\cdot))$. Thus, for any status quo state, $\omega = (\tilde{G}, (G, S))$, contained in $H_i$ the probability that the coming state, $\omega' = (\tilde{G}', (G', S'))$, will be contained in $H_i$ is 1. Formally, we have $H_i \in \mathcal{L}^*$ and thus for all $\omega = (\tilde{G}, (G, S)) \in H_i$

$$p^*(H_i | \tilde{G}, (G, S)) := \varepsilon(H_i(G', S')|G', S') \rho(d(G', S')|\omega, \sigma^*(\tilde{G}, (G, S))) = 1$$

where

$$H_i(G', S') := \left\{ \tilde{G}' \in \mathcal{G}_{D(G')} : (\tilde{G}', (G', S')) \in H_i \right\}.$$

Let $H_i(\omega) \subset \mathcal{G}^m$ be a subset of proposal $m$-tuples such that if the status quo state $\omega$ is in $H_i$, then for each $G_D \in H_i(\omega)$

$$\rho(D_{g \times F}(H_i)|\omega, G_D) = 1$$

where $D_{g \times F}(H_i)$ is the $(g \times F)$-domain of the absorbing set $H_i \in \mathcal{L}^*$ given by

$$D_{g \times F}(H_i) := \left\{ (G, S) \in g \times F : (\tilde{G}, (G, S)) \in H_i \text{ for some } \tilde{G} \in \mathcal{G}_{D(G)} \right\}.$$

Thus, if the status quo is $\omega \in H_i$ and if $m$-tuple $G_D \in H_i(\omega)$ is proposed by the players, then the coming regular state, $(G', S')$ will be contained in the $(g \times F)$-domain, $D_{g \times F}(H_i)$, of the absorbing set $H_i \in \mathcal{L}^*$ with probability 1. But now because $H_i \in \mathcal{L}^*$ is absorbing for the equilibrium transition, $p^*(\cdot | \cdot)$, and because $(\omega, G_D) \in grH_i(\cdot) \subset H_i \times \mathcal{G}^m$, for each $(G', S') \in D_{g \times F}(H_i),$

$$\varepsilon(H_i(G', S')|G', S') = 1.$$

Because $H_i$ is absorbing for the equilibrium transition, $p^*(\cdot | \cdot)$, and because the equilibrium transition, $p^*(\cdot | \cdot)$, is strategically informed, meaning that

$$p^*(\cdot | \cdot) = q(\cdot | \cdot, \sigma^*(\cdot))$$

where $\sigma^*(\cdot)$ is an equilibrium stationary Markov strategy profile, we know that there exists for each basin, $H_i$, such a collection of $m$-tuples of risky network proposals, $H_i(\omega)$, such that

$$\sigma^*(H_i(\omega)|\omega) = \sigma^*(H_i(\tilde{G}, (G, S))|\tilde{G}, (G, S)) = 1 \text{ for all } \omega := (\tilde{G}, (G, S)) \in H_i.$$

Finally, we know that for all $(G', S') \in D_{g \times F}(E),$

$$\varepsilon(H_i(G', S')|G', S') = 1,$$

simply because for each $(G', S') \in D_{g \times F}(E), (\tilde{G}', (G', S')) \in H_i$ for all $\tilde{G}' \in H_i(G', S')$ and $p^*(H_i|\omega) = q(H_i|\omega, \sigma^*(\omega)) = 1$ for all $\omega \in H_i.
In the long run, we know that with probability 1 the equilibrium network-coalition formation process
\[
\left\{ \tilde{W}_n^* \right\}_{n=1}^{\infty} = \left\{ (\tilde{G}_n^*, (G_n^*, S_n^*)) \right\}_{n=1}^{\infty}
\]
governed by the equilibrium Markov transition \( p^* (\cdot | \cdot) = q (\cdot | \cdot, \sigma^* (\cdot)) \) will enter one of the strategic basins of attraction, \( H_i \in \{ H_1, \ldots, H_N \} \) and will remain there. Thus, given any strategic basin of attraction \( H_i \) any subset of risky networks,
\[
H_i (\omega) := H_i (\tilde{G}, (G, S)) \subset G \text{ such that }
\]
\[
\sigma^* (H_i (\omega) | \omega) := \sigma^* (H_i (\tilde{G}, (G, S)) | \tilde{G}, (G, S)) = 1 \text{ for all } \omega := (\tilde{G}, (G, S)) \in H_i,
\]
is strategically stable for the players.
References


